METTI5 Tutorial T1 on

"Experimental identification of low order model"

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Abstract

The goal of this tutorial is to apply the system identification technique in order to obtain an accurate direct model devoted to measurements inversion. This tutorial is closely related to Lesson L8. A simple experiment will be used in order to give the basic ideas of the optimal experiment in system identification approach. It will be particularly emphasized on the choice of the excitation sequence. Two methods will be used: the correlation method and the parametric method. A direct model will thus be obtained and it will be analyzed in terms of reliability and accuracy.

As a conclusion, it will be pointed out the advantages of this approach with respect to the classical one based on the resolution of the heat diffusion equation.

1 System description

1.1 Real life context

In high enthalpy plasma flows for aerospace applications but also in high power pulsed laser physics as well as in laser surface hardening monitoring, very high heat fluxes on the order of several MW/m^2 have to be measured (see Figure 1). Only transient measurement techniques have been developed so far. Recently, fast transient heat flux measurements have been conducted using a novel calibration approach. In principle, these sensors can stay a very short time, of the order of milliseconds, in the harsh environment in order not to reach melting temperature. The transient response of the sensor recorded during this short time is used to estimate the heat flux.

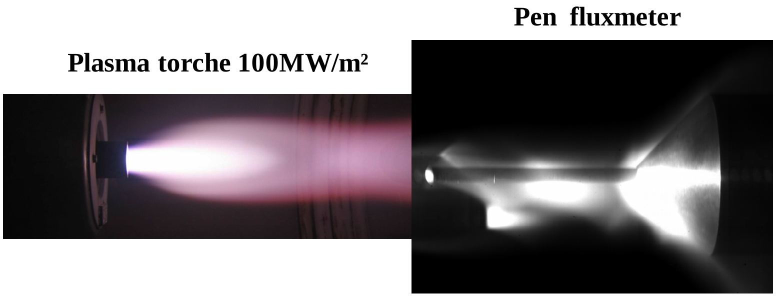


Figure 1: heat flux measurement in a plasma flow using a fluxmeter.

One basic principle of such a heat flux sensor is to measure the temperature, usually with a thermocouple embedded inside an appropriate material which is part of the sensor, and to estimate the heat flux from inverse heat conduction calculation. Consequently, highest reliability in terms of measurement accuracy and local resolution within the sensor is reached when the distance between the temperature measurement and the heated surface is small (see Figure 2). Solving the inverse heat conduction problem requires a model, so-called direct model (DM) of the heat transfer from the heated surface of the sensor to the location of the thermocouple inside the sensor. Model presented in the literature generally assume that the sensor behaves like a semi-infinite wall.

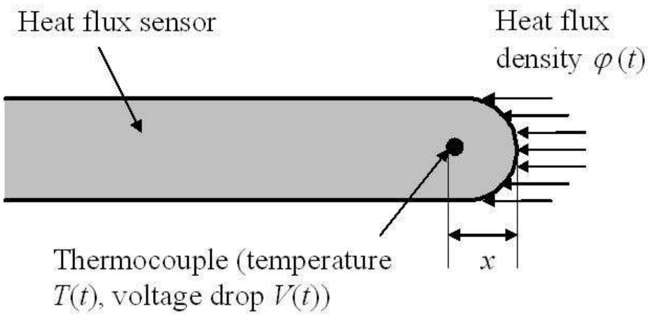


Figure 2: fluxmeter description

Considering a particular sensor and linear heat transfer, i.e. constant material properties for the measurement time, the Duhamel's theorem makes sure that the DM can be viewed as the impulse response. This is the transient temperature of the thermocouple due to a heat flux on the form of a Dirac function. Assuming the sensor behaves as a semi infinite wall, the impulse response is analytically expressed according to the thermal properties of the medium as well as the location of the thermocouple inside the sensor. However, in real configuration, the heat flux sensors involve several types of materials arranged in a more or less complicated way. The knowledge of their thermophysical properties as well as the thermal contact resistances at the interfaces have to be known if one wants to use the finite element method to solve a detailed DM. Furthermore, it appears that the thermocouple rise time can be greater than the sampling time interval during the acquisition process. This means that the thermocouple junction is not at a uniform temperature for each acquisition time. In other words, there exists a temperature gradient in the junction and one must also consider heat diffusion inside the junction. Obviously, it is not allowed applying the electrothermal conversion between the voltage drop and the temperature of the thermocouple since it rests on the assumption of a uniform temperature of the junction.

In order to overcome these problems, the basic idea rests on the system identification of the heat flux sensor. It consists in calibrating the sensor by applying a measurable transient heat flux in the time domain of interest using a modulated laser source. Given that the identified system has to be accurate for all the swept frequencies, it is searched a heat flux waveform whose Power Spectral Density is comparable to that of a white noise. This calibration thus involves the realization of a specific experimental setup in laboratory which allows applying the heat flux on the form of a random or Pseudo Random Binary Signal (PRBS) and measuring it precisely as well as the voltage drop at the thermocouple. This approach does not require knowing the thermophysical properties of materials as well as the exact location of the thermocouple. Also the knowledge of the thermocouple inertia is circumvented since it is taken into account within the calibration. From a theoretical point of view, the identified system is the direct model when solving the inverse heat conduction problem. It means that the identified system and the voltage drop measurement at the sensor during the use of the sensor for a given application is sufficient to estimate the heat flux. This approach does not depend on a particular sensor geometry which then allows manufacturing also particular sensor geometries for particular applications. Major drawback of this approach is that the calibration must be realized in the same conditions than those encountered on the process during the use of the heat flux sensor otherwise a linear behavior has to be assumed. In other words, one must reproduces in the laboratory the same boundary conditions in terms of transient and magnitude of heat flux.

1.2 System identification hardware

The sensor consists of a cylindrical copper tube, where a thermocouple is integrated. The tip is of spherical form. The thermocouple is of type K with a junction diameter of 0.08 mm. As specified by the manufacturer, the rise time for the thermocouple is about 120 msec. However, nowadays the heat flux sensor has to be used on comparable time duration, but with 0.1 msec sampling interval.

The Figure 3 shows the schematic view of the “scale 1” experimental setup for these calibration measurements. Since the null-point calorimeter is used to measure very high heat flux densities up to 100 MW/m^2, the laser pulse energy has to be high in order to achieve a sufficiently resolvable signal of the thermocouple. The laser in use is a laser diode that can provide 2000 W at 980 nm wavelength. It is known in system identification, that the best results are achieved when laser pulses of variable length are generated in order to better distinguish each of the characteristic times of the system: the diffusion in the tip, the response of the thermocouple and the influence of the interfaces. The laser is driven using a function generator that can generate a Pseudo Random Binary Signal (PRBS) for example. On the Figure 4, the Power Spectral Density (PSD) of such a signal is compared to that of a

Dirac function and that of the Heaviside function. It is obviously found that the best representation of the frequencies is given with a perfect Dirac function which is very difficult to implement experimentally, particularly in terms of reproducibility. Concerning the two others, the best one is the PRBS because its PSD is superior to the Heaviside function PSD in the studied frequency range.

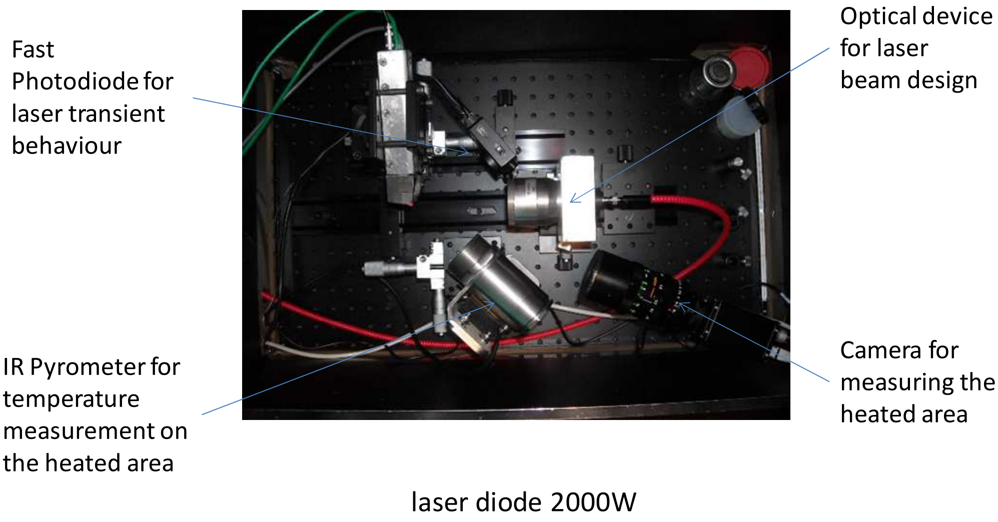


Figure 3: system identification hardware at scale 1

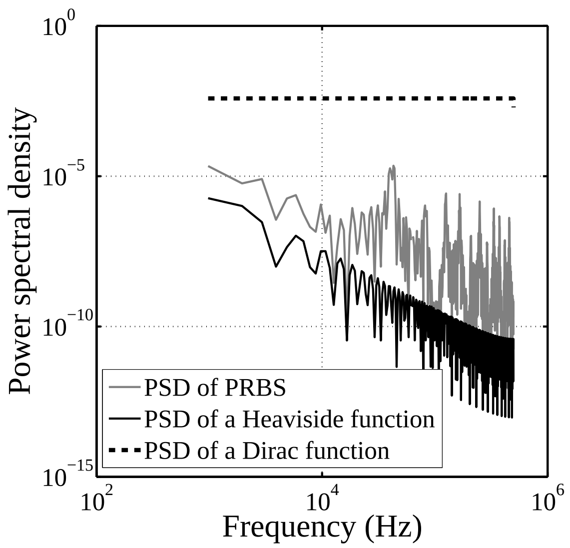


Figure 4: Power Spectral Density of a Dirac function, a Heaviside function and a PRBS.

A very small part (less than 2%) of the heat flux from the laser is recorded using a fast photodiode,. All data are recorded using a fast oscilloscope. Moreover, the thermocouple data is amplified with a constant gain of 250. The incident heat flux density for the laser is deduced from the calibration curve of the manufacturer based on blackbody absorption. The sensor is made off copper which has been oxidized in a furnace at 400°C during four hours in order to achieve high surface catalycity and to reach high absorptivity. Its emissivity has then been measured to 0.7 at the laser wavelength.

The noise measurement at the thermocouple is recorded without heating the sensor by the laser and results (sampling time is 100 µsec) and the noise histogram taht shows that the noise has a Gaussian distribution are presented in Figure 5. On the other hand, the computation of the noise auto-correlation function is represented in Figure 6. This function is close to 0 from the second point and is thus equivalent to a Dirac function. In conclusion, all the assumptions concerning noise measurement have been checked and the application of the least square algorithm in order to estimate the parameters of the non integer system is fully consistent.

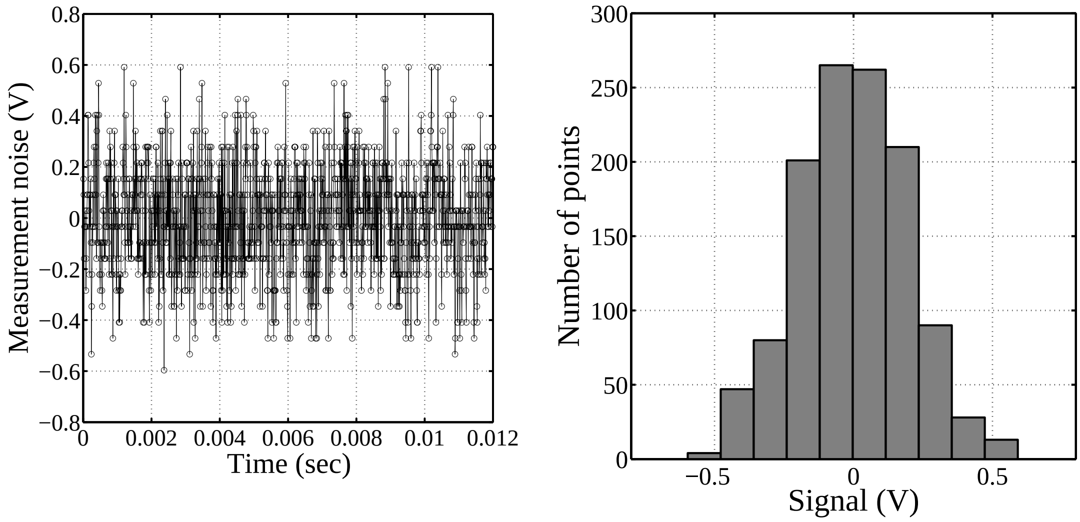


Figure 5: Noise measurement and noise distribution function.

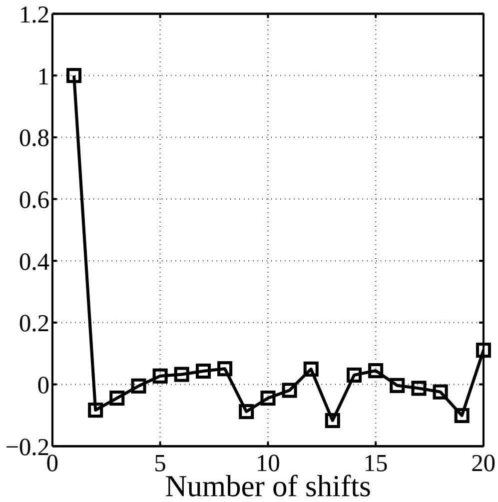


Figure 6: Autocorrelation function applied to the noise measurement.

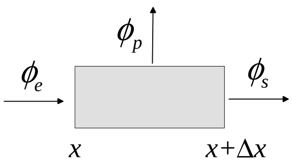
The uncertainty on the emissivity measurement is about 6%. Concerning the laser heat flux measured with a fan cooled broad band sensor, the constructor gives a calibration certificate and a relative precision of about 1%. Finally, the radius of the beam laser is measured manually and the uncertainty is estimated to be 5%. In conclusion, the uncertainties of the measured heat flux density absorbed by the sensor is approximately 17% which is high compared to the uncertainties involved by the parameter estimation method.

For this tutorial it has been realized an experimental setup at scale 1/400. Indeed, the laser diode is 5W maximum power. It is driven through National Instrument card under Labview software. The other parts of the experiment remain identical to those of the experiment scale 1.

1.3 Heat transfer model in the fluxmeter

Thermal properties of the fin are λ for the thermal conductivity, α for the thermal diffusivity, ρ for the density and C_p for the specific heat. Heat losses between the fin and the surrounding fluid are characterized by the heat exchange coefficient *h*. The section of the thin is denoted *S* and the circumference is denoted *s*. If the temperature depends only on the longitudinal direction, (Bi = h d/λ << 0.1) carrying out a heat flux balance on a slice of the fin of width Δx, we get:

$$\phi_e - \phi_s - \phi_p = \rho C_p S \Delta x \frac{\mathrm{d}T}{\mathrm{d}t} \tag{1}$$



with:

$$\phi_e = -\lambda S \frac{\mathrm{d}T(x,t)}{\mathrm{d}x} \tag{2}$$

$$\phi_s = -\lambda S \frac{\mathrm{d}T(x+\Delta x,t)}{\mathrm{d}x} \tag{3}$$

$$\phi_p = h s \Delta x \left(T(x,t) - T_\infty\right) \tag{4}$$

Substituting these relations into (1) we obtain:

$$\lambda S \left(\frac{\mathrm{d}T(x+\Delta x,t)}{\mathrm{d}x} - \frac{\mathrm{d}T(x,t)}{\mathrm{d}x} \right) - h s \Delta x \left(T(x,t) - T_\infty\right) = \rho C_p S \Delta x \frac{\mathrm{d}T(x,t)}{\mathrm{d}t} \tag{5}$$

that is,

$$\lambda S \frac{\mathrm{d}^2 T(x,t)}{\mathrm{d}x^2} \Delta x - h s \Delta x \left(T(x,t) - T_\infty\right) = \rho C_p S \Delta x \frac{\mathrm{d}T(x,t)}{\mathrm{d}t} \tag{6}$$

Putting $T'(x) = T(x) - T_\infty$, the above expression becomes:

$$\lambda S \frac{\mathrm{d}^2 T'(x,t)}{\mathrm{d}x^2} - h s T'(x,t) = \rho C_p S \frac{\mathrm{d}T'(x,t)}{\mathrm{d}t} \tag{7}$$

Applying the Laplace transform to T', we obtain:

$$\frac{\mathrm{d}^2\theta(x,p)}{\mathrm{d}x^2} - k^2 \theta(x,p) = 0 \tag{8}$$

with

$$k^2 = \frac{\rho C_p S p + h s}{\lambda S} = \frac{p}{\alpha} + \frac{h s}{\lambda S} \tag{9}$$

The solution is:

$$\theta(x,p) = A \exp(k x) + B \exp(-k x) \tag{10}$$

When $x \to \infty$ we find A=0. Then:

$$\theta_0 = L\left(T(0,t) - T_\infty\right) = B \tag{11}$$

and thus:

$$\theta(x,p) = \theta_0 \exp(-k x) \tag{12}$$

The heat dissipated by the semi-infinite fin is:

$$\psi_0(p) = -\lambda S \left.\frac{\mathrm{d}\theta(x,p)}{\mathrm{d}x}\right|_{x=0} \tag{13}$$

that is,

$$\theta(x,p) = \frac{\psi_0(p)}{\lambda S k} \tag{14}$$

Relation (14) shows that thermal impedance of the fin is:

$$Z_a = \frac{\exp(-k x)}{\lambda S k}, k = \sqrt{\frac{p}{\alpha} + \frac{h s}{\lambda S}} \tag{15}$$

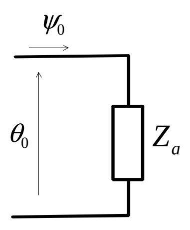


Figure 7: thermal impedance of the fin.

2 Correlation and spectral techniques

2.1 Theoretical background

Based on the linearity assumption, the measured temperature $\langle y_0(t)\rangle$ is related to the heat flux $\varphi_0(t)$ from the convolution product:

$$\langle y_0(t)\rangle = \int_0^\infty h_0(t-\tau)\varphi_0(\tau)\,\mathrm{d}\tau + e(t) \tag{16}$$

$h_0(t)$ is the impulse response and $e(t)$ denotes the measurement error.

Now, let us multiply the two members of this equality by $\varphi_0(t-\tau)$ and integrate from t=0 to infinity. It is then obtained:

$$\int_0^\infty \langle y_0(t)\rangle \varphi_0(t-\tau)\,\mathrm{d}t = \int_0^\infty h_0(t-\tau)\left(\int_0^\infty \varphi_0(\tau)\varphi_0(t-\tau)\,\mathrm{d}t\right)\mathrm{d}\tau + \int_0^\infty \varphi_0(t-\tau)e(t)\,\mathrm{d}t \tag{17}$$

One recognizes the cross and auto correlation functions, $C_{y_0\varphi}(\tau)$, $C_{\varphi e}(\tau)$ and $C_{\varphi\varphi}(\tau)$. Relation (17) is thus written on the following form:

$$C_{y_0\varphi}(\tau) = \int_0^\infty h_0(t-\tau)C_{\varphi\varphi}(\tau)\,\mathrm{d}\tau + C_{\varphi e}(\tau) \tag{18}$$

By choosing the heat flux $\varphi_0(t)$ as a white noise lead to:

$$C_{\varphi\varphi}(\tau) = \delta(\tau) \tag{19}$$

Finally, if it is admitted that the noise measurement is not correlated to the heat flux ($C_{e\varphi} = 0$), relation (18) is summarized to:

$$C_{y_0\varphi}(\tau) = h_0(\tau) \tag{20}$$

One sees that the impulse response is equal to the cross correlation function between the temperature of the sensor and the heat flux. This approach is very sensitive to noise measurement magnitude, so one would rather use the power spectral density (PSD) instead of the correlation function. In practice, it consists in applying the Fourier transform on the cross correlation and auto correlation functions, i. e.:

$$\mathrm{FFT}\left[C_{y_0\varphi}(\tau)\right] = \mathrm{FFT}\left[\int_0^\infty h_0(t-\tau)C_{\varphi\varphi}(\tau)\,\mathrm{d}\tau\right] = Y_0(f)\Phi_0(f) = S_{y_0\varphi}(f) \tag{21}$$

and

$$\mathrm{FFT}\left[C_{\varphi\varphi}(\tau)\right] = \mathrm{FFT}\left[\int_0^\infty \varphi_0(t-\tau)\varphi_0(\tau)\,\mathrm{d}\tau\right] = \Phi_0(f)^2 = S_{\varphi\varphi}(f) \tag{22}$$

$Y_0(f)$ and $\Phi_0(f)$ are the Fourier transforms of the temperature and the heat flux respectively as well as $S_{\varphi\varphi}(f)$ and $S_{y_0\varphi}(f)$ are the auto and cross PSD. Then, by applying the Fourier transform on relation (18) it is immediately obtained:

$$S_{y_0\varphi}(f) = H(f)S_{\varphi\varphi}(f) + S_{\varphi e}(f) \tag{23}$$

Finally, assuming that the noise measurement is not correlated with the heat flux ($S_{\varphi e}(f) = 0$), the expression of the transfer function is:

$$H(f) = \frac{S_{y_0\varphi}(f)}{S_{\varphi\varphi}(f)} \tag{24}$$

Since the length of the experiment is set to a fixed value τ, the real input signal is:

$$\varphi_\Pi(t) = \varphi_0(t)\Pi_\tau(t) \tag{25}$$

In this relation, $\Pi_\tau(t) = 1$ when $0 \le t \le \tau$ and 0 elsewhere. Then applying the Fourier transform on the heat flux leads to:

$$\Phi_\Pi(f) = \Phi_0(f) * \left(\tau \frac{\sin(\pi\tau f)}{\pi\tau f}\right) \tag{26}$$

It appears that the Fourier transform of the heat flux is convoluted by the sinus cardinal function. Usually, the heat flux is pre windowed by a specific function $g_\tau(t)$ which decreases the influence of the function $\Pi_\tau(t)$ as:

$$\varphi_\Pi(t) = \varphi_0(t)g_\tau(t) \tag{27}$$

For example, it is often used of the Hanning window defined by:

$$g_\tau(t) = 0.5\left(1 - \cos\left(\frac{2\pi t}{\tau}\right)\right) \tag{28}$$

It is used an improved estimation of $S_{y_0\varphi}(f)$ and $S_{\varphi\varphi}(f)$ proposed by Welch. The method consists in dividing the time series data into possible overlapping segments, computing the auto and cross power spectral densities and averaging the estimates.

2.2 Application

During the tutorial we will generate 2 kinds of heat flux sequence:

2.2.1 linear swept-frequency cosine signal

We will consider first a heat flux on the form of a linear swept-frequency cosine signal:

$$\varphi_0(t) = \sin\left(2\pi f(t) t\right) \tag{29}$$

The frequency varies linearly with time as:

$$f(t) = f_0 + \frac{f_1 - f_0}{t_1} t, \; 0 \le t \le t_1 \tag{30}$$

f_0 is the instantaneous frequency at time 0, and f_1 is the instantaneous frequency at time t_1 (see Figure 8).

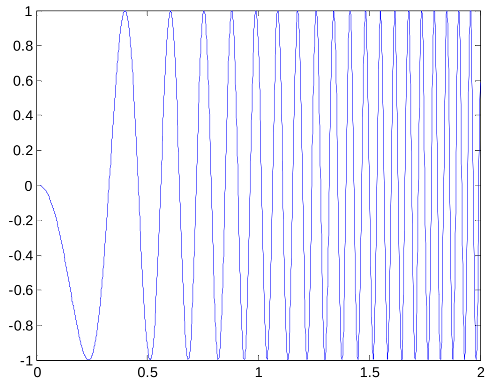


Figure 8: example of linear swept-frequency cosine signal with $f_0 = 0.1\,\mathrm{Hz}$ and $f_1 = 10\,\mathrm{Hz}$.

It will be used the Welch technique. The swept-frequency cosine heat flux waveform has two major interesting features. The first is that the offset must be easily removed from the experimental heat flux in order to fully satisfy the relation (19). The second feature is that the explored frequential domain, defined from the sensitivity analysis, is perfectly swept. Furthermore, the use of auto and cross power spectral density functions allows defining the so called coherence function as:

$$C_{y_0\varphi}(f) = \frac{\left|S_{y_0\varphi}(f)\right|^2}{S_{y_0y_0}(f) S_{\varphi\varphi}(f)} \tag{31}$$

This function can be viewed as the correlation coefficient between the temperature and the heat flux and lies between 0 and 1. If it is 1 at a certain frequency, then there is perfect correlation between the two signals at this frequency. In other words, there is consequently no noise interfering at this frequency, what lead to:

$$S_{\varphi e}(f) = S_{y_0y_0}(f)\left(1 - C_{y_0\varphi}(f)\right) \tag{32}$$

2.2.2 PRBS signal

In a second stage we will consider the heat flux sequence as a Pseudo Random Binary Signal (PRBS). "White noise" is the term given to completely random unpredictable noise, such as the hiss you hear on an untuned radio. It has the property of having components at every frequency. A pseudo-random binary sequence (PRBS) can also have this property, but is entirely predictable. A PRBS is rather like a long recurring decimal number- it looks random if you examine a short piece of the sequence, but it actually repeats itself every *m* bits. Of course, the larger *m* is, the more random it looks. You can generate a PRBS with a shift register and an XOR gate. Connecting the outputs of two stages of the shift register to the XOR gate, and then feeding the result back into the input of the shift register will generate a PRBS of some sort. Some combinations of outputs produce longer PRBSs than others- the longest ones are called m-sequences (where m means "maximum length"). A binary sequence (BS) is a sequence of *N* bits,

$$a_j \text{ for } j = 0,1,\ldots,N-1,$$

i.e. m ones and $N - m$ zeros. A BS is pseudo-random (PRBS) if its autocorrelation function:

$$C(v) = \sum_{j=0}^{N-1} a_j \, a_{j+v} \tag{33}$$

has only two values:

$$C(v) = \begin{cases} m, \text{ if } v \equiv 0 \pmod N \\ m \times c, \text{otherwise} \end{cases} \tag{34}$$

where

$$c = \frac{m-1}{N-1} \tag{35}$$

is called the duty cycle of the PRBS.

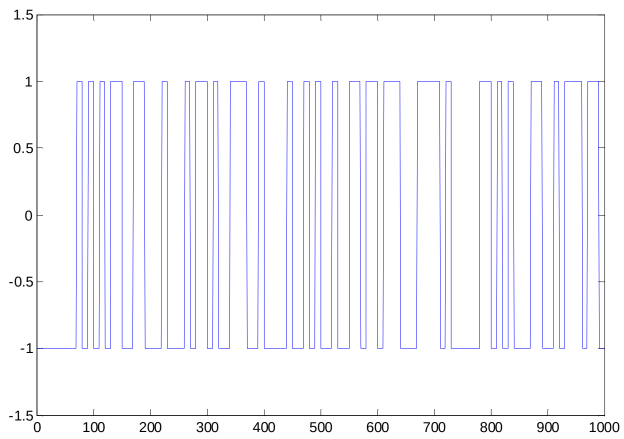


Figure 9: example of a PRBS sequence (X axis is the number of samples).

A PRBS is random in a sense that the value of an a_j element is independent of the values of any of the other elements, similar to real random sequences.

It is 'pseudo' because it is deterministic and after N elements it starts to repeat itself, unlike real random sequences, such as sequences generated by radioactive decay or by white noise. The PRBS is more general than the n-sequence, which is a special pseudo-random binary sequence of n bits generated as the output of a linear shift register. An n-sequence always has a 1/2 duty cycle and its number of elements $N = 2^k - 1$.

3 Parametric identification

3.1 AR models, theoretical background

As expressed by relation (15) heat transfer in a fin is modelled as:

$$\theta(x,p) = \frac{\exp(-k\,x)}{\lambda S k}\psi(p)\text{, with } k = \sqrt{\frac{p}{\alpha} + \frac{h s}{\lambda S}} \tag{36}$$

We can write that:

$$k = \sqrt{\frac{p}{\alpha} + \frac{h s}{\lambda S}} = \sqrt{\frac{h s}{\lambda S}}\sqrt{\frac{\lambda S}{\alpha h s}\,p + 1} \tag{37}$$

Using the series it is found:

$$k = \sqrt{\frac{h s}{\lambda S}}\sqrt{\frac{\lambda S}{\alpha h s}\,p + 1} = \sqrt{\frac{h s}{\lambda S}}\sum_{n=0}^{\infty}\frac{(2n-1)!}{(2n)!}\left[\frac{1}{2}\frac{\lambda S}{\alpha h s}\,p\right]^n \tag{38}$$

On the other hand one has:

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!},\quad \forall z \tag{39}$$

Replacing k in the exponential function and this former with its series, it is found that on obtain an equivalent expression of relation (36) on the following form:

$$\sum_{n=0}^{\infty}\alpha_n\, s^{n+1}\,\theta_m(s) = \sum_{n=0}^{\infty}\beta_n\, s^n\,\Phi(s) \tag{40}$$

Where α_n and β_n have complex but analytical expressions.

Given to the initial condition (temperature is zero at each point of the domain) and using the property:

$$L\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}\frac{\mathrm{d}^n f(0)}{\mathrm{d}t^n} \tag{41}$$

it thus appear that relation (40) is equivalent to:

$$\sum_{n=0}^{\infty}\alpha_n \frac{\mathrm{d}^{n+1}T\left(M,t\right)}{\mathrm{d}t} = \sum_{n=0}^{\infty}\beta_n \frac{\mathrm{d}^{n}\varphi\left(t\right)}{\mathrm{d}t} \tag{42}$$

Using the discrete form of the derivatives an equivalent form of relation (42) that lead to express the temperature at time $k\,\Delta t$ from the heat flux and the temperature at previous times as:

$$T_k = b_0\,\varphi_{k-nk} + b_1\,\varphi_{k-nk-1} + \cdots + b_{nb}\,\varphi_{k-nk-nb} - a_1\,T_{k-1} - \cdots - a_{na}T_{k-na}\,,\quad k=\left(1,\cdots,N\right) \tag{43}$$

In this equation the continuous time t have been discretized in N sampling intervals of length Δt, such as $T = N\,\Delta t$, where T is the final time. Parameters $\left(a_1, a_2, \cdots, a_{na}, b_0, b_1 \cdots, b_{nb}\right)$ are real numbers. The integer nk corresponds to the lag time that depends on the distance between the output and the input. This lagging effect only depends on the distance between the tip of the insert and the point M.

Since the white noise term $e\left(k\right)$ defined by

$$e_k = T_k - Y_k \tag{44}$$

enters as a direct error in the difference equation, the model (43) is often called an equation error model.

By substituting equation (44) in equation (43) one obtains

$$Y_k = \mathbf{H}_k\,\Theta + e_k \tag{45}$$

where

$$\mathbf{H}_k = \left[-Y_{k-1} \quad -Y_{k-2} \quad \cdots \quad -Y_{k-na} \quad \varphi_{k-nk} \quad \varphi_{k-nk-1} \quad \cdots \quad \varphi_{k-nk-nb}\right] \tag{46}$$

is the regression matrix and

$$\Theta = \left[a_1 \quad a_2 \quad \cdots \quad a_{na} \quad b_0 \quad \cdots \quad b_{nb}\right] \tag{47}$$

is the vector of unknowns.

By performing N successive measurements, relation (45) can be written on a matrix form as

$$\mathbf{Y} = \mathbf{H}\,\boldsymbol{\Theta} + \mathbf{E} \tag{48}$$

where

$$\mathbf{Y}^T = \left[Y_{nd} \quad Y_{nd+1} \quad \cdots \quad Y_N\right] \tag{49}$$

$$\mathbf{E}^T = \left[e_{nd} \quad e_{nd+1} \quad \cdots \quad e_N\right] \tag{50}$$

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{nd} \\ \mathbf{H}_{nd+1} \\ \vdots \\ \mathbf{H}_N \end{bmatrix} \tag{51}$$

and

$$nd = \max\left(na+1, nb+nk+1\right) \tag{52}$$

Resolution of equation (48) in the least square sense leads to

$$\hat{\mathbf{\Theta}} = \left(\mathbf{H}^T\,\mathbf{H}\right)^{-1}\mathbf{H}^T\,\mathbf{Y} \tag{53}$$

The choice of $\mathbf{\Lambda} = [na, nb, nk]$ is made by collecting in a matrix all the values of $\mathbf{\Lambda}$ to be investigated and looking on the value of the Aikake criterion defined by

$$\Psi = \frac{1+n/N}{1-n/N}V\,,\;\; n = na+nb+1 \tag{54}$$

where *n* is the total number of estimated parameters and *V* is the loss function defined by

$$V = \sum_{k=1}^{N} {e_k}^2 \tag{55}$$

Standard errors of the estimates are calculated from the covariance matrix of $\hat{\mathbf{\Theta}}$. If the assumptions of additive, zeros mean, constant variance σ^2 and uncorrelated errors are verified, the covariance matrix is expressed as

$$\operatorname{cov}\left(\hat{\mathbf{\Theta}}\right) = \left(\mathbf{H}^T\,\mathbf{H}\right)^{-1}\sigma^2 \tag{56}$$

An estimate of the variance σ^2, denoted s^2, is

$$s^2 = \frac{1}{N-n}\mathbf{E}^T\,\mathbf{E} \tag{57}$$

From equation (53), by substituting the estimated vector $\hat{\mathbf{\Theta}}$ in equation (48), one finds

$$\hat{\mathbf{\Theta}} = \mathbf{\Theta} + \left(\mathbf{H}^T\,\mathbf{H}\right)^{-1}\mathbf{H}^T\,\mathbf{E} \tag{58}$$

Thereby, asymptotic estimation of the mean of $\hat{\mathbf{\Theta}}$ is

$$E\left[\hat{\mathbf{\Theta}}\right] = \mathbf{\Theta} + \left(E\left[{\mathbf{H}_k}^T\,\mathbf{H}_k\right]\right)^{-1}E\left[{\mathbf{H}_k}^T\,e_k\right] \tag{59}$$

From this last relation it is clear that the estimation is unbiased if e_k is uncorrelated with φ_k or if $E[e_k] = 0$. Thereby, the typical whiteness test is to determine the covariance estimate

$$R_{ek} = \frac{1}{N}\sum_{r=0}^{N} e_{r-k}\, e_r \tag{60}$$

Independence between *e* and *q* is tested using the sample covariance

$$R_{e\varphi k} = \frac{1}{N}\sum_{r=0}^{N} e_{r-k}\, \varphi_r \tag{61}$$

It may be noted that when using a model as equation (43), the least square procedure automatically makes the correlation between e_k and $\varphi_{k-\delta}$ zero for $\delta = nk, nk+1, \cdots, nk+n_b-1$. In practice, relations (60) and (61) are computed using the Fourier Transform of the convolution product.

3.2 Application

The method will be applied during the tutorial starting from the response to a PRBS sequence for the heat flux.

3.3 NI models, theoretical background

At very short times, heat losses are negligible face to the heat diffusion in the fin. Therefore:

$$\frac{p}{\alpha} \gg \frac{h\,s}{\lambda S}\, and\ k \approx \sqrt{\frac{p}{\alpha}} \tag{62}$$

Since:

$$\theta(x,p) = \frac{\sum_{n=0}^{\infty}\frac{(-k\,x)^n}{n!}}{\lambda S\, k}\psi(p)\text{, with } k = \sqrt{\frac{p}{\alpha}+\frac{h\,s}{\lambda S}} \tag{63}$$

Replacing *k* in this relation leads to:

$$\theta(x,p) = \frac{\sum_{n=0}^{\infty}\frac{-x^n}{\alpha^{n/2}\,n!}\,p^{n/2}}{\sqrt{\lambda\,\rho\,C_p}\,S\,p^{1/2}}\psi(p) \tag{64}$$

That can be also written as:

$$\sqrt{\lambda\,\rho\,C_p}\,S\,p^{1/2}\,\theta(x,p) = \sum_{n=0}^{\infty}\frac{-x^n}{\alpha^{n/2}\,n!}\,p^{n/2}\,\psi(p) \tag{65}$$

Since we have shown in L8 course that relation:

$$L\left(\frac{\mathrm{d}^{\nu} f(t)}{\mathrm{d}t^{n}}\right) = s^{\nu}\,F(s) - \sum_{k=0}^{n-1} s^{n-k-1}\frac{\mathrm{d}^{\nu} f(0)}{\mathrm{d}t^{\nu}} \tag{66}$$

remains true even if ν is a real number or more generally complex, we can express relation (65) in the time domain as:

$$\alpha_0 \, \mathrm{D}^{n/2}\{T_m(t)\} = \sum_{n=0}^{\infty} \beta_n \, \mathrm{D}^{n/2}\{\varphi(t)\} \tag{67}$$

With:

$$\alpha_0 = \sqrt{\lambda \rho C_p} \, S \ \text{ and } \ \beta_n = \frac{-x^n}{\alpha^{n/2} \, n!} \tag{68}$$

It appears thus than model (42) will not be accurate enough to describe the transient behaviour at the short times. As said in lesson 8, an optimal structure of a low order model for heat transfer problem by diffusion must be of the following form:

$$\sum_{n=0}^{\infty} \alpha_n \, \mathrm{D}^{n/2}\{T_m(t)\} = \sum_{n=0}^{\infty} \beta_n \, \mathrm{D}^{n/2}\{\varphi(t)\} \tag{69}$$

3.4 Application

During the tutorial, we will apply the NISI method to estimate parameters in relation (69) in order to fit temperature measurements at the sensor starting from a PRBS of the heat flux.