EUROTHERM Seminar 94
Advanced Spring School
Thermal Measurements & Inverse Techniques
Station Biologique de ROSCOFF
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L9 Part B
Thermal characterization of homogeneous and heterogeneous materials using SVD techniques

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$$\mathbf{Y} + \boldsymbol{\varepsilon} = \mathbf{M}\boldsymbol{\theta}$$

Noise-corrupted observers

Sensitivity matrix

Badly conditioned

Noise amplifier

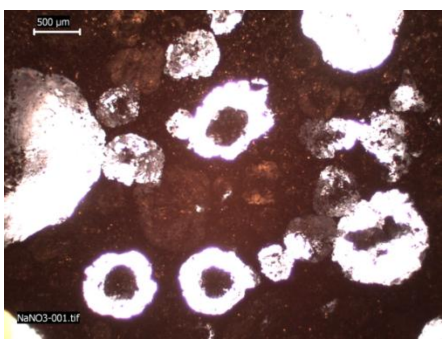
SVD of the observed thermal response

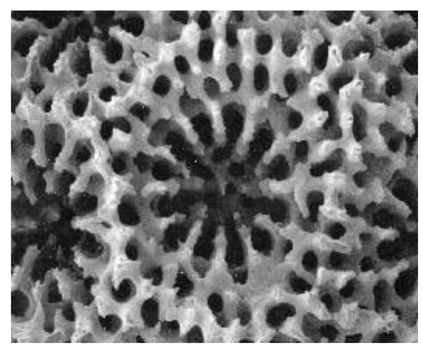
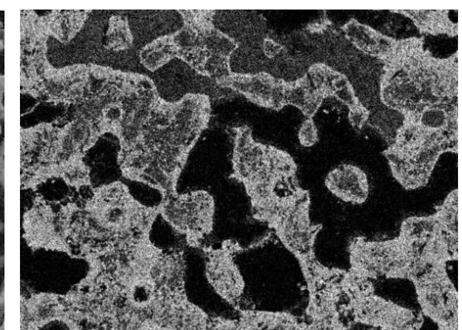
REGULARIZATION
Experiment design
SVD of **M**
Cost function
etc.

Materials & Objectives

Homogeneous materials
Diffusivity tensor

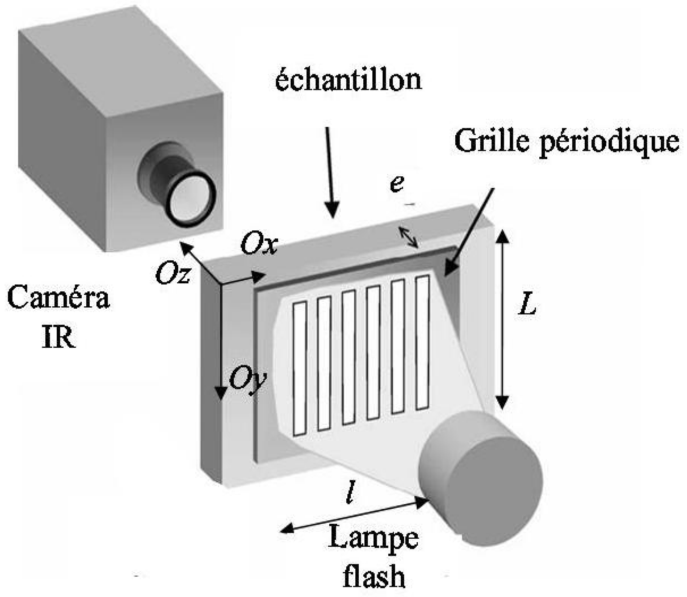
Multi-phase materials
Phases discrimination & interfaces location
Diffusivity of the phases
Thermal resistance at the interfaces

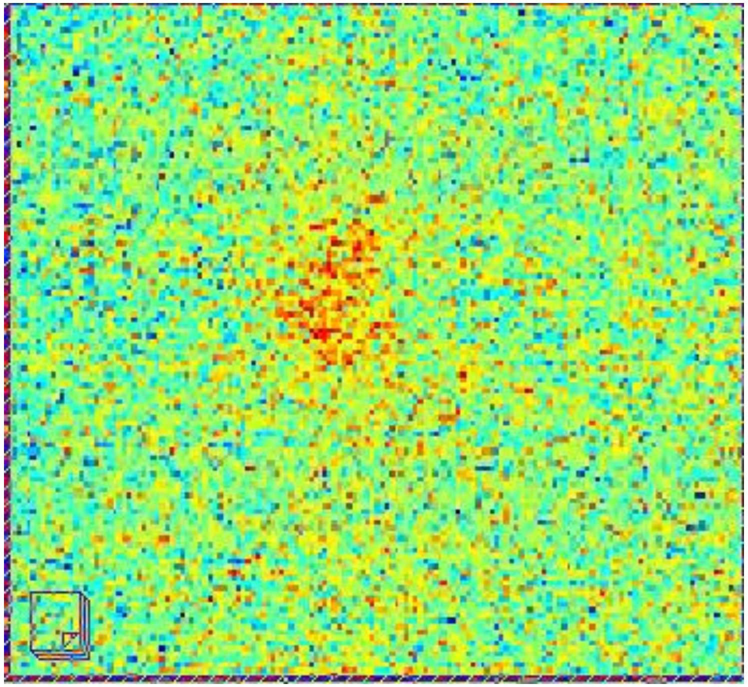


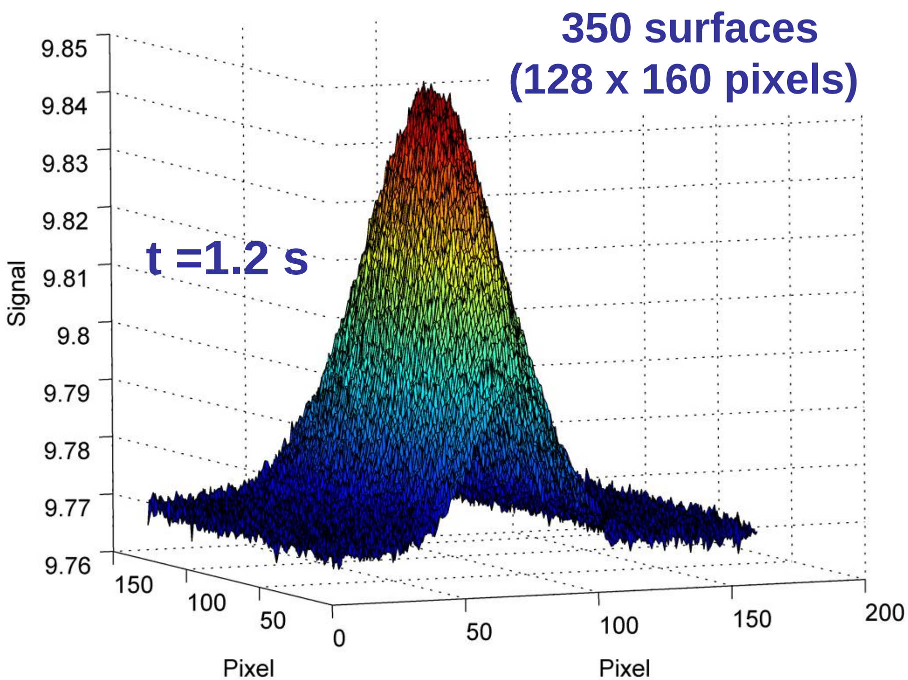
Scales: µm – cm
Thin samples - Infrared thermography

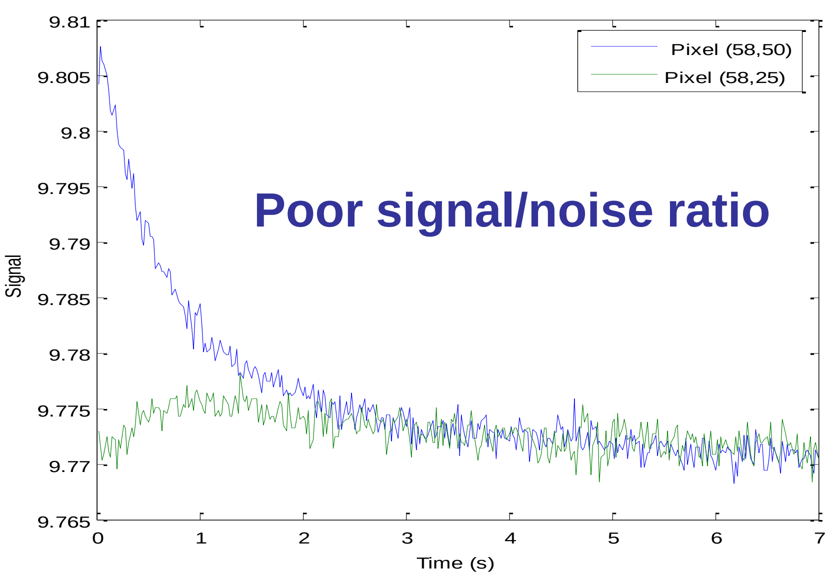
Experimental framework





- Thin sample
- Photo-thermal excitation (i.e. laser, lamp):
  - Time: modulated, flash, step …
  - Space: spot, motif, random …
- Thermal response observation (rear or front face) by an IR camera equipped by a focal point array





HIGH DIMENSION

1 map = 250 x 250 pixels = 62 500 data
1 exp. > 1000 maps
Number of data > **60 millions of data !**

POOR QUALITY DATA

Linearity -> Low amplitudes
Noise -> Optics, resolution, env.

Analytical solutions
Homogeneous materials
Experimental conditions

SVD-based methods
Dimension reduction
Noise effects reduction
Negligible bias

SVD of the thermal field

Proposed for the first time in 1901by Pearson, re-invented many times during the XX century

- Principal components analysis (PCA)
- Singular values decomposition (SVD)
- Karhunen-Loève decomposition (KLD)
- Proper orthogonal decomposition (POD)
- Hotteling transformation
- Empirical eigenfunctions
- ….

Application fields

- Data analysis (graphics, statistics …)
- Forms recognition, faults detection …
- Models reduction (turbulence)
- …

A tool providing optimal low-dimensional approximations of high-dimensional (infinite-dimensional) problems

$$T(\mathbf{x},t)$$

THERMAL FIELD

space-time dependent function in a bounded region verifying

$$\forall t, \quad \int_{\Omega} T^2(\mathbf{x},t)\,dx < \infty$$

THERMAL FIELD - SVD

Projection on the orthogonal basis defined by the spectral decomposition of the ENERGY FUNCTION

$$W(\mathbf{x},\mathbf{x}') \equiv \int_t T(\mathbf{x},t)T(\mathbf{x}',t)dt = \sum_{m=1}^{\infty} \sigma_m^2 V_m(\mathbf{x})V_m(\mathbf{x}')$$

$$\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq 0$$

Total energy $\int_\Omega W(\mathbf{x},\mathbf{x})d\mathbf{x} = \sum_{m=1}^{\infty} \sigma_m^2$

$$\boxed{\forall t,\ T(\mathbf{x},t) = \sum_{m=1}^{\infty} V_m(\mathbf{x}) z_m(t)}$$

Orthogonal eigenfunctions

$$\langle V_k, V_m \rangle_\Omega \equiv \int_\Omega V_k(\mathbf{x})V_m(\mathbf{x})d\mathbf{x} = \delta_{km}$$

Orthogonal states

$$\langle z_m(t), z_k(t) \rangle_t \equiv \int_t z_m(t)z_k(t)dt = \delta_{mk}\sigma_m^2$$

Optimal/closest linear *r*-dimensional approximations

Best approximation in the sens of unitarily invariant norms

$$\boxed{\forall t,\ T_r(\mathbf{x},t) = \sum_{m=1}^{r} V_m(\mathbf{x}) z_m(t)} \longrightarrow \| e_r \|^2 \equiv \int_t \int_\Omega e_r(\mathbf{x},t)dxdt = \sum_{m=r+1}^{\infty} \sigma_m^2$$

$$\forall t,\ e_r(\mathbf{x},t) \equiv T(\mathbf{x},t) - T_r(\mathbf{x},t)$$

An excellent tool for data reduction

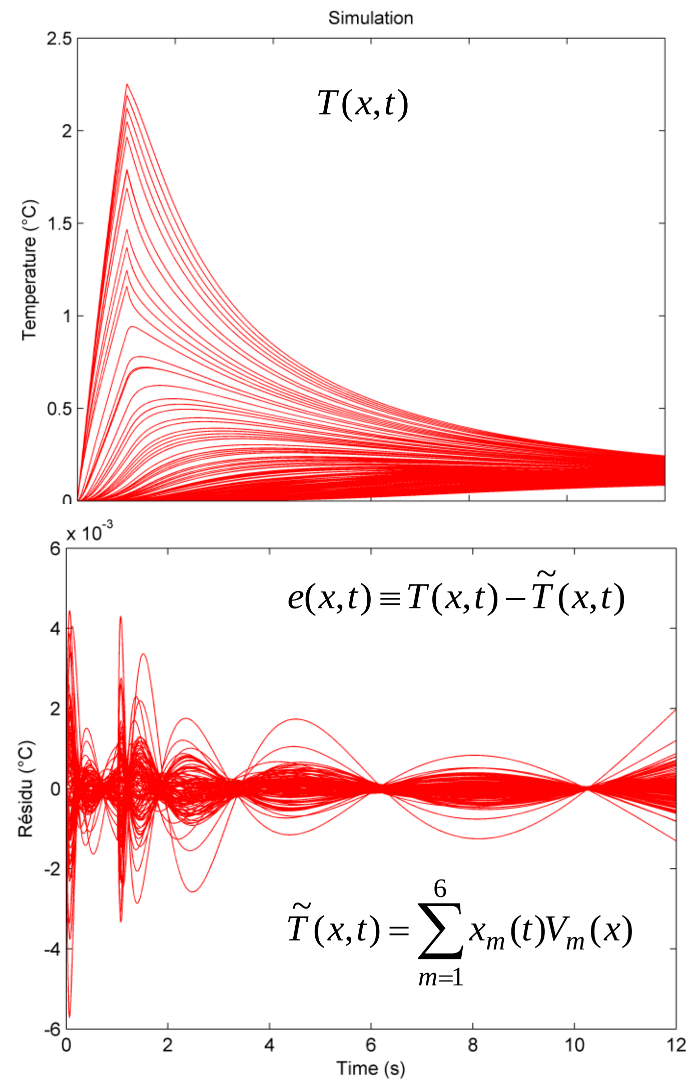
High quality approximations of the thermal field can often be reached with a very low number of eigenelements

EXAMPLE
1 map = 250 x 250 = 62500 pixels
1000 sampling times

1000 maps x 62500 pixels/map =
= 62.5 millions data

> 99% reduction

6 maps of 62500 pixels
+ 6 time series of length 1000
= 380 000 data



Thermal response of a thin plate (6mm x 6mm) excited by a laser spot during 1s

Noise propagation through SVD

$$\tilde{T}(\mathbf{x},t) = T(\mathbf{x},t) + \varepsilon(\mathbf{x},t)$$

$$\forall \mathbf{x},\mathbf{x}' \quad W_\varepsilon(\mathbf{x},\mathbf{x}') \equiv \int_t \varepsilon(\mathbf{x},t)\varepsilon(\mathbf{x}',t)dt = \sigma_\varepsilon^2 \delta(\mathbf{x}-\mathbf{x}')$$

(1) spatially-uncorrelated noise has no effect on eigenfunctions, the noise being entirely reported on states

$$\tilde{W}(\mathbf{x},\mathbf{x}') = W(\mathbf{x},\mathbf{x}') + W_\varepsilon(\mathbf{x},\mathbf{x}') = \sum_{m=1}^{\infty} V_m(\mathbf{x})\tilde{\sigma}_m^2 V_m(\mathbf{x}') \quad with \quad \tilde{\sigma}_m^2 = \sigma_m^2 + \sigma_\varepsilon^2 \delta(\mathbf{x}-\mathbf{x}')$$

$$\boxed{\tilde{T}(\mathbf{x},t) = \sum_{m=1}^{\infty} V_m(\mathbf{x})\tilde{z}_m(t)}$$

$$\tilde{z}_m(t) = \int_\Omega \tilde{T}(\mathbf{x},t)V_m(\mathbf{x})d\mathbf{x} = \int_\Omega T(\mathbf{x},t)V_m(\mathbf{x})dx + \int_\Omega \varepsilon(\mathbf{x},t)V_m(\mathbf{x})dx = z_m(t) + \underline{\gamma_m(t)}$$

$$\forall m, \quad \sigma_{\varepsilon,m}^2 \equiv \int_t \gamma_m^2(t)dt = \sigma_\varepsilon^2$$

$$noise/signal = \sigma_\varepsilon^2 / \sigma_m^2$$

$$I(\mathbf{x}) \equiv \frac{\int_t T^2(\mathbf{x},t)\,dt}{\int_t \varepsilon^2(\mathbf{x},t)\,dt} = \frac{\sum_{m=1}^{\infty} V_m^2(\mathbf{x})\sigma_m^2}{\sum_{m=1}^{\infty} V_m^2(\mathbf{x})\sigma_\varepsilon^2}$$

$$I_r(\mathbf{x}) = \frac{\sum_{m=1}^{r} V_m^2(\mathbf{x})\sigma_m^2}{\sum_{m=1}^{r} V_m^2(\mathbf{x})\sigma_\varepsilon^2}$$

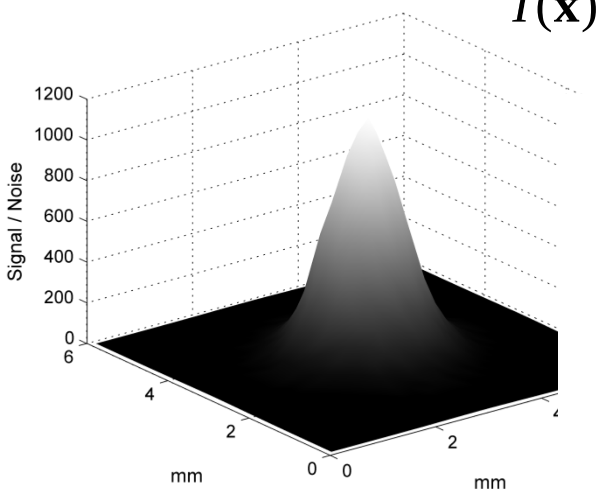
(2) SVD truncation acts as a signal/noise ratio amplifier

$$\boxed{\frac{I(\mathbf{x})}{I_r(\mathbf{x})} = \underbrace{\left(1 + \frac{\sum_{m=r+1}^{\infty} V_m^2(\mathbf{x})\sigma_m^2}{\sum_{m=1}^{r} V_m^2(\mathbf{x})\sigma_m^2}\right)}_{O(1)} \left(\frac{\sum_{m=1}^{r} V_m^2(\mathbf{x})}{\sum_{m=1}^{\infty} V_m^2(\mathbf{x})}\right) \approx \left(\frac{\sum_{m=1}^{r} V_m^2(\mathbf{x})}{\sum_{m=1}^{\infty} V_m^2(\mathbf{x})}\right) < 1}$$

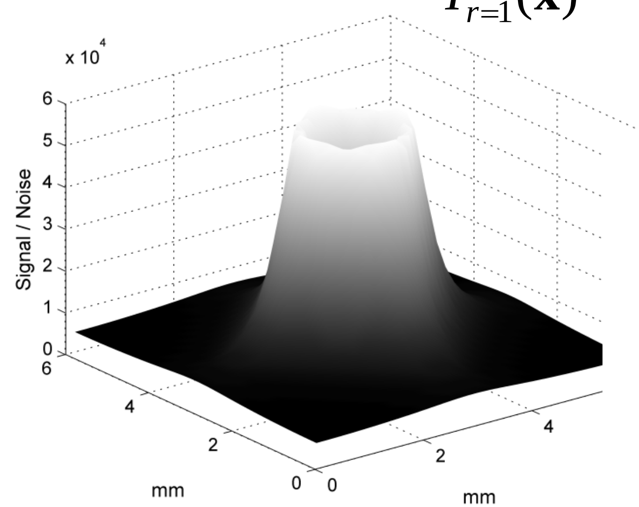
$$\sigma_1^2 >> \sigma_2^2 >> \cdots \geq 0$$

Signal/noise amplification is as much significant as the dimension of the approximation reduces

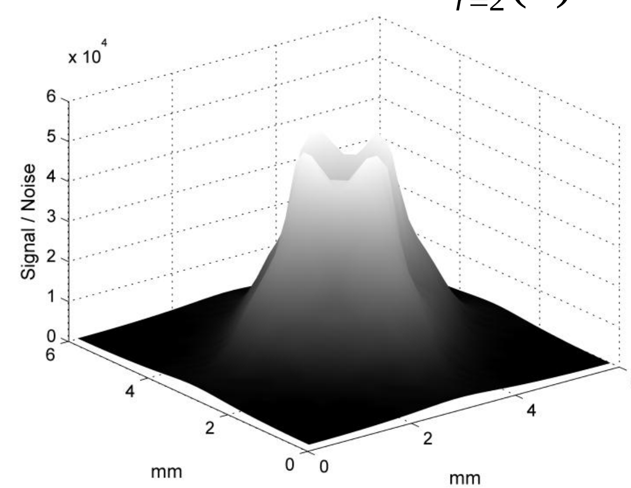
$I(\mathbf{x})$



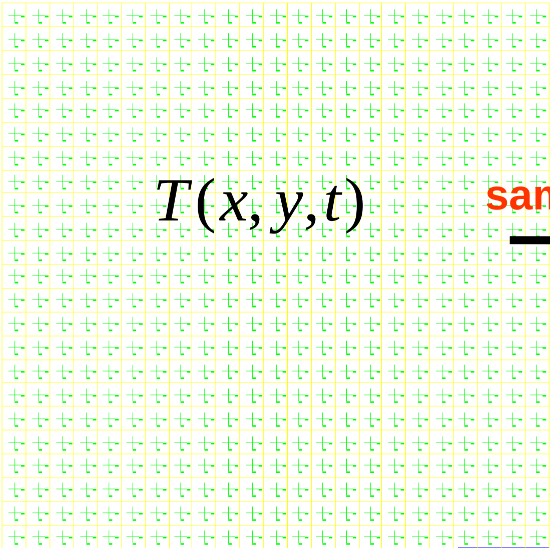
$I_{r=1}(\mathbf{x})$



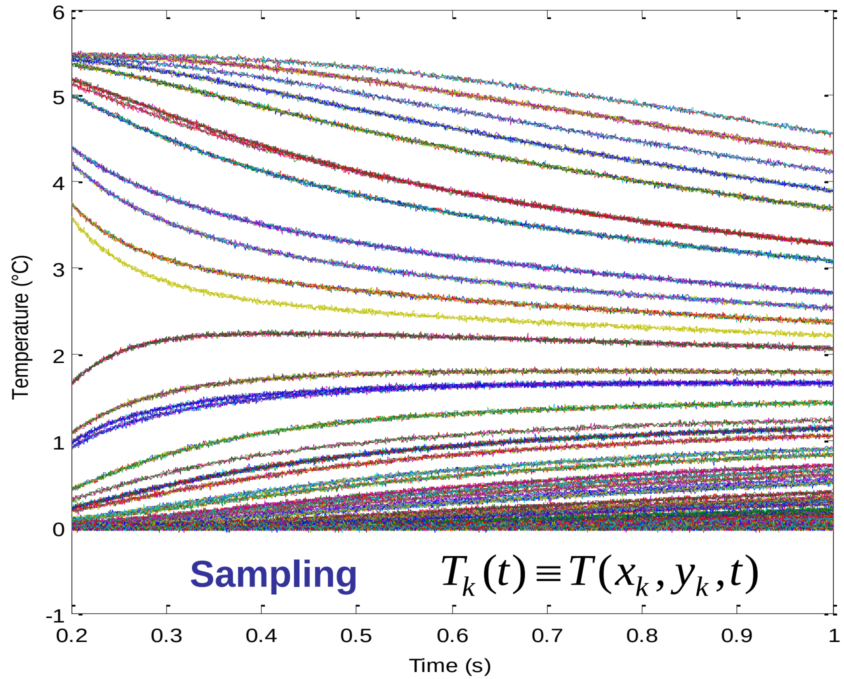
$I_{r=2}(\mathbf{x})$



SVD in practice



sampling



Time-dependent vector of high dimension (n x1)

$$\mathbf{T}(t) = \begin{bmatrix} T_1(t) \\ T_2(t) \\ \vdots \\ T_n(t) \end{bmatrix}$$

Energy matrix (n x n)
& Spectral decomposition

$$\mathbf{W} = \int_{t=t_o}^{t_f} \mathbf{T}(t)\mathbf{T}^t(t)dt$$

$$\mathbf{W} = \mathbf{V}\boldsymbol{\Sigma}\,\mathbf{V}^t$$

Matrix of eigenfunctions $V_m(x,y)$ column-wise placed

Diagonal matrix of eigen-values $\sigma_1^2 > \sigma_2^2 > \ldots > 0$

Vector of states

$$\mathbf{Z}(t) \equiv \begin{bmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{bmatrix} = \mathbf{V}^t\mathbf{T}(t)$$

Homogeneous materials

$$\Omega:\ 0 < x < L_x,\ 0 < y < L_y$$

Photo-thermal excitation (i.e. laser, lamp)
Time: modulated, flash, step …
Space: spot, motif, random …

Photo thermal perturbation – Establishment of a initial temperature field for further estimations

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha_x \frac{\partial^2 T(x,y,t)}{\partial x^2} + \alpha_y \frac{\partial^2 T(x,y,t)}{\partial y^2} - \beta T(x,y,t) + \varphi(x,y,t)$$

$$\left.\frac{\partial T(x,y,t)}{\partial x}\right|_{x=0,L_x} = 0, \qquad \left.\frac{\partial T(x,y,t)}{\partial y}\right|_{y=0,L_y} = 0 \qquad T(x,y,0) = 0$$

Equations governing thermal relaxation of the initial temperature field

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha_x \frac{\partial^2 T(x,y,t)}{\partial x^2} + \alpha_y \frac{\partial^2 T(x,y,t)}{\partial y^2} - \beta T(x,y,t)$$

$$\left.\frac{\partial T(x,y,t)}{\partial x}\right|_{x=0,L_x} = 0, \qquad \left.\frac{\partial T(x,y,t)}{\partial y}\right|_{y=0,L_y} = 0 \qquad T(x,y,0) = T_o(x,y)$$

ESTIMATION

α_x, α_y, β

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha_x \frac{\partial^2 T(x,y,t)}{\partial x^2} + \alpha_y \frac{\partial^2 T(x,y,t)}{\partial y^2} - \beta T(x,y,t)$$

+ adiabatic boundary conditions

$$T(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) z_m(t)$$

$$\langle V_k, V_m \rangle_\Omega \equiv \int_\Omega V_k(x) V_m(x) dx = \delta_{km}$$

$$\langle z_m(t), z_k(t) \rangle_t \equiv \int_t z_m(t) z_k(t) dt = \delta_{mk} \sigma_m^2$$

$$\frac{1}{\sigma_k^2} \left\langle \frac{dz_i(t)}{dt}, z_k(t) \right\rangle_t = \frac{1}{\sigma_i^2} \left\langle \frac{dz_k(t)}{dt}, z_i(t) \right\rangle_t$$

Integration over Ω

$$\frac{d\langle T(x,y,t) \rangle_\Omega}{dt} = -\beta \langle T(x,y,t) \rangle_\Omega$$

ESTIMATION β

linear least squares methods

$$\left\langle \langle \circ, V_k \rangle_\Omega, z_i(t) \right\rangle_t$$

$$i = 1, 2, 3, \cdots; \quad k = i, i+1, i+2, \cdots$$

$$y_{ik} = \alpha_x \left\langle \frac{\partial^2 V_i(x,y)}{\partial x^2}, V_k(x,y) \right\rangle_\Omega + \alpha_y \left\langle \frac{\partial^2 V_i(x,y)}{\partial y^2}, V_k(x,y) \right\rangle_\Omega$$

$$y_{ik} = \frac{z_i(t_f) z_k(t_f) - z_i(0) z_k(0)}{\sigma_i^2 + \sigma_k^2} + \beta \delta_{ik}$$

How many equations, which ones ?

$$y_{ik} = \frac{z_i(t_f)z_k(t_f) - z_i(0)z_k(0)}{\sigma_i^2 + \sigma_k^2} + \beta\delta_{ik}$$

$$\tilde{z}_{i,k}(t_f) = z_{i,k}(t_f) + \varepsilon_{i,k}(t_f)$$

$$\tilde{z}_{i,k}(0) = z_{i,k}(0) + \varepsilon_{i,k}(0)$$

$$\tilde{\sigma}_i^2 = \sigma_i^2 + \sigma_\varepsilon^2$$

$$\tilde{\beta} = \beta + \varepsilon_\beta$$

$$\boxed{\tilde{y}_{ik} = y_{ik}\frac{1}{1+2\left(\frac{\sigma_\varepsilon^2}{\sigma_i^2+\sigma_k^2}\right)} + \frac{f(noise_z)}{\sigma_i^2+\sigma_k^2+2\sigma_\varepsilon^2} + \varepsilon_\beta\delta_{ik}}$$

$$E[\tilde{y}_{ik}] = y_{ik}\frac{1}{1+2\left(\frac{\sigma_\varepsilon^2}{\sigma_i^2+\sigma_k^2}\right)}$$

bias

$$\mathrm{var}_y \equiv E[(\tilde{y}_{ik} - y_{ik})^2] =$$

$$= \frac{z_k^2(t_f) + z_i^2(t_f) + z_k^2(0) + z_i^2(0)}{(\sigma_i^2+\sigma_k^2+2\sigma_\varepsilon^2)^2}\mathrm{var}_\varepsilon + \mathrm{var}_\beta\,\delta_{ik}$$

$$\sigma_1^2 \ge \sigma_2^2 \ge \cdots \ge 0 \qquad \sigma_1^2 >> \sigma_2^2 >> \cdots \ge 0$$

$$\sigma_\varepsilon^2 << \sigma_i^2 + \sigma_k^2$$

Ordering by decreasing $\sigma_i^2 + \sigma_k^2$

y_{11}

y_{12}

Cut-off

y_{22}

$\vdots$

Adding a more one = Increasing bias

Two unknown parameters –
Two equations for estimations

$$y_{11} = \alpha_x \left\langle \frac{\partial^2 V_1(x,y)}{\partial x^2}, V_1(x,y) \right\rangle_\Omega + \alpha_y \left\langle \frac{\partial^2 V_1(x,y)}{\partial y^2}, V_1(x,y) \right\rangle_\Omega$$

$$y_{12} = \alpha_x \left\langle \frac{\partial^2 V_1(x,y)}{\partial x^2}, V_2(x,y) \right\rangle_\Omega + \alpha_y \left\langle \frac{\partial^2 V_1(x,y)}{\partial y^2}, V_2(x,y) \right\rangle_\Omega$$

(α_x, α_y) **ESTIMATE**

$$\boxed{\begin{bmatrix} \hat{\alpha}_x \\ \hat{\alpha}_y \end{bmatrix} = \mathbf{M}^{-1}\mathbf{y}}$$

with:

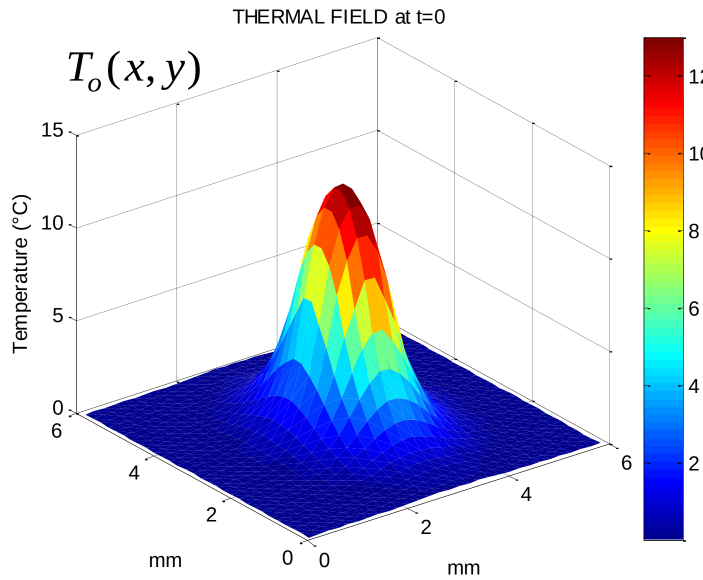
$$\left\{ \begin{array}{l} \mathbf{y} = \begin{bmatrix} \Delta\tilde{z}_1\tilde{z}_1 / (\tilde{\sigma}_1^2 + \tilde{\sigma}_1^2) + \hat{\beta} \\ \Delta\tilde{z}_1\tilde{z}_2 / (\tilde{\sigma}_1^2 + \tilde{\sigma}_2^2) \end{bmatrix} \\ \\ \mathbf{M} = \begin{bmatrix} \left\langle \partial_x^2 V_1(x,y), V_1(x,y) \right\rangle_\Omega & \left\langle \partial_y^2 V_1(x,y), V_1(x,y) \right\rangle_\Omega \\ \left\langle \partial_x^2 V_1(x,y), V_2(x,y) \right\rangle_\Omega & \left\langle \partial_y^2 V_1(x,y), V_2(x,y) \right\rangle_\Omega \end{bmatrix} \end{array} \right.$$

$$\mathrm{E}[\hat{\alpha}_x] = \alpha_x; \quad \mathrm{E}[\hat{\alpha}_y] = \alpha_y$$

provided that: $\sigma_\varepsilon^2 << \sigma_1^2 + \sigma_2^2$

NUMERICAL EXAMPLES

Noise amplitude: 0.02, 0.1°C, 0.5°C,1°C



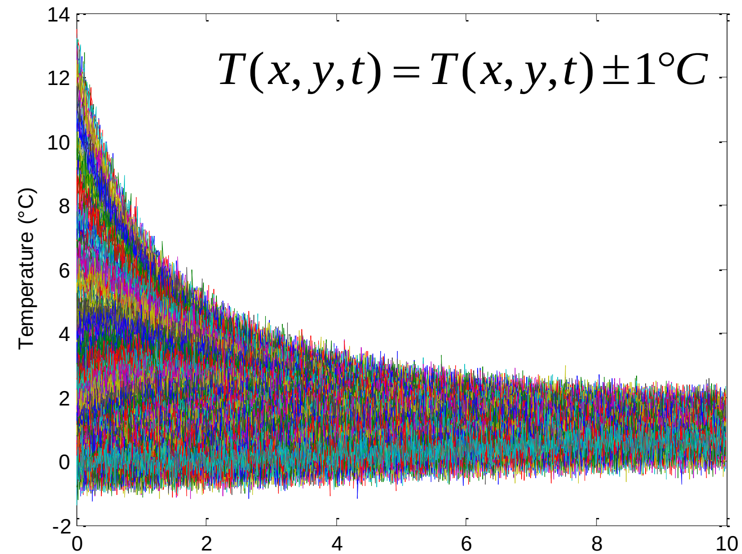
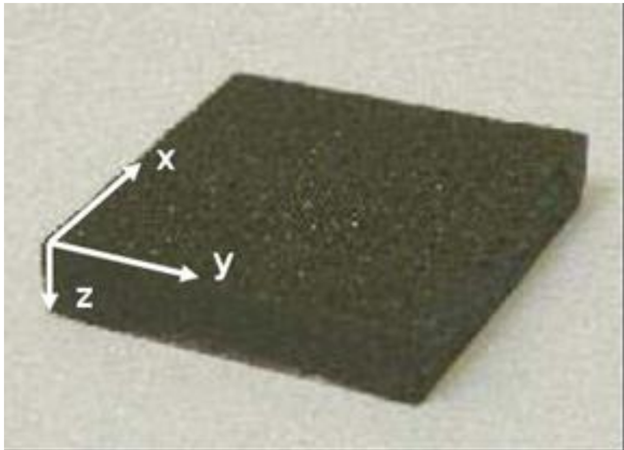


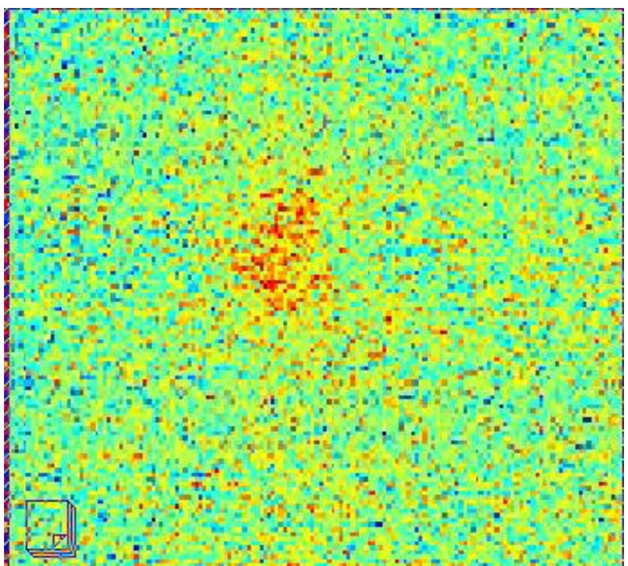
Table 3. Estimated values for thermal parameters. True values are $\beta = 0.152 s^{-1}$, $\alpha_x = 0.1515 \times 10^{-6}\, m^2.s^{-1}$ and $\alpha_y = 0.3030 \times 10^{-6}\, m^2.s^{-1}$.

Noise ampl. (°C)	$\hat{\beta}$ (s^{-1})	$\left\lvert \frac{\beta - \hat{\beta}}{\beta} \right\rvert \times 100$	$\hat{\alpha}_x$ $(\times 10^{-6} m^2 s^{-1})$	$\left\lvert \frac{\alpha_x - \hat{\alpha}_x}{\alpha_x} \right\rvert \times 100$	$\hat{\alpha}_y$ $(\times 10^{-6} m^2 s^{-1})$	$\left\lvert \frac{\alpha_y - \hat{\alpha}_y}{\alpha_y} \right\rvert \times 100$
± 1.00	0.0152 (±0.5x10^{-3})	0.055	0.1572 (± 0.02)	3.72	0.2889 (±0.036)	4.66
± 0.50	0.0152 (±0.2x10^{-3})	0.06	0.1540 (± 0.007)	1.62	0.2989 (±0.0162)	1.37
± 0.10	0.0152 (±0.04x10^{-3})	0.002	0.1517 (± 0.0015)	0.12	0.3026 (±0.0037)	0.15
± 0.02	0.0152 (±0.01x10^{-3})	0.0007	0.1515 (± 0.0003)	0.022	0.3029 (±0.0007)	0.0003

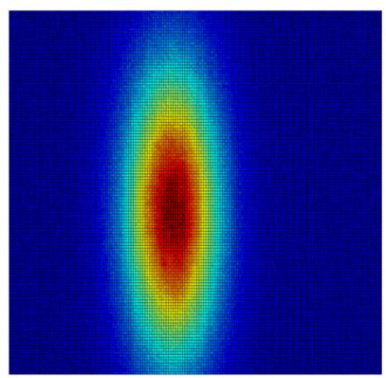
EXPERIMENTAL TEST
Carbon/Epoxy composite material



	ICAM CETHIL	TREFLE
α_x (m^2/s)	4×10^{-7}	3.80×10^{-7}
α_y (m^2/s)	3.5×10^{-6}	3.54×10^{-6}



ORTOTROPE

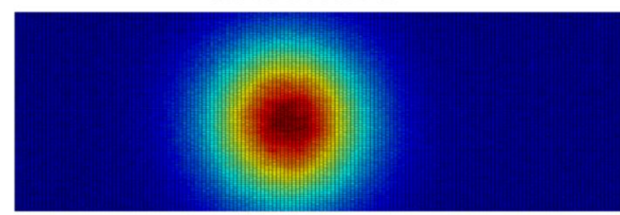


$$x \leftarrow x\left(\frac{\alpha}{\alpha_x}\right)^{1/2} \; ; \; y \leftarrow y\left(\frac{\alpha}{\alpha_y}\right)^{1/2}$$

$$\alpha = \alpha_x^{1/2}\alpha_y^{1/2}$$

Ok

As ISOTROPE



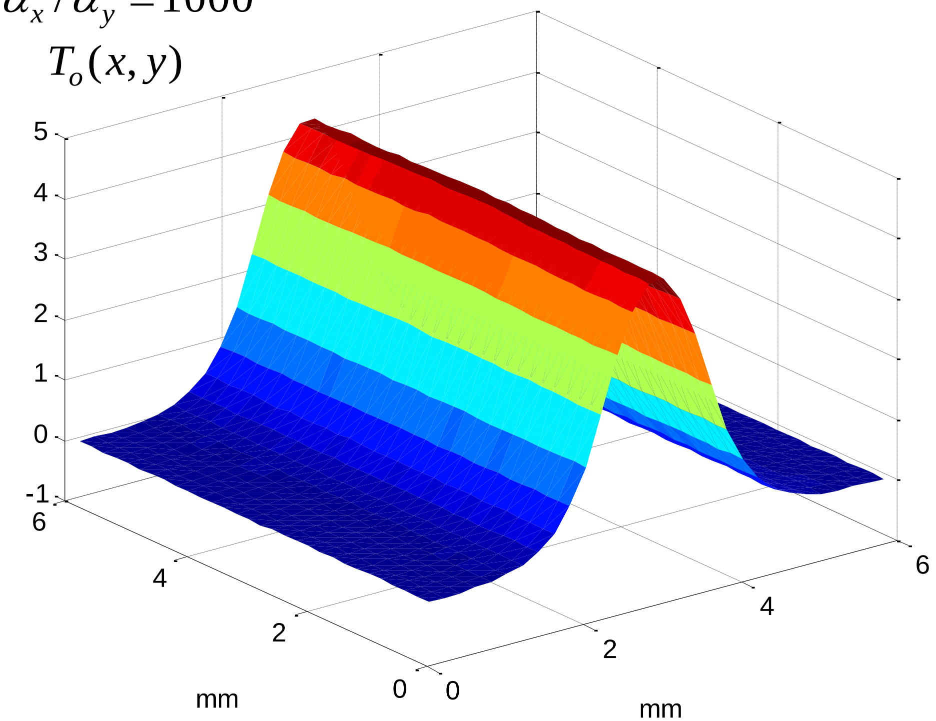
$\beta = 0.152 s^{-1}$ $\alpha_x = 1.515\times10^{-7}\ m^2.s^{-1}$ $\alpha_y = 3.030\times10^{-7}\ m^2.s^{-1}$

Many other numerical tests for reliability analysis

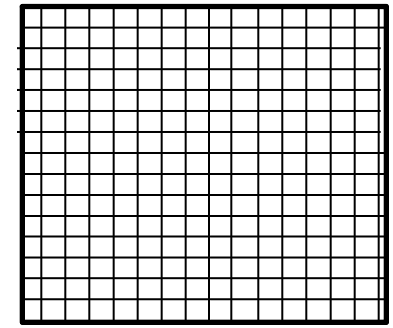
- Much more high beta (x100)
- Much more high diffusivities (x1000)
- Much more low diffusivities (/1000)
- Much more high α_x/α_y (~100, ~1000)
- Much more low α_x/α_y

$\alpha_x / \alpha_y = 1000$

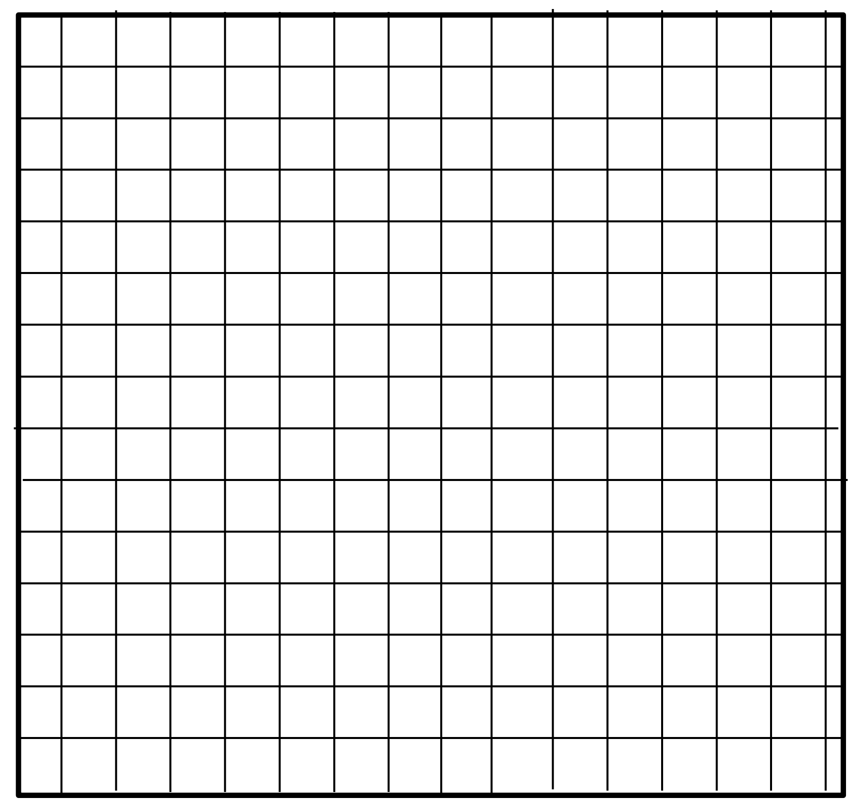
$T_o(x, y)$



Higher resolution - Smaller area



Greater area – Lower resolution



Many other numerical tests for reliability analysis

Much more high beta (x100)
Much more high diffusivities (x1000)
Much more low diffusivities (/1000)
Much more high α_x/α_y (~100, ~1000)
Much more low α_x/α_y (i.e. $\alpha_x/\alpha_y = 1.01$)

Table 3. Estimated values for thermal parameters. True values are $\beta = 0.152 s^{-1}$, $\alpha_x = 0.1515\times10^{-6}\ m^2.s^{-1}$ and $\alpha_y = 0.3030\times10^{-6}\ m^2.s^{-1}$.

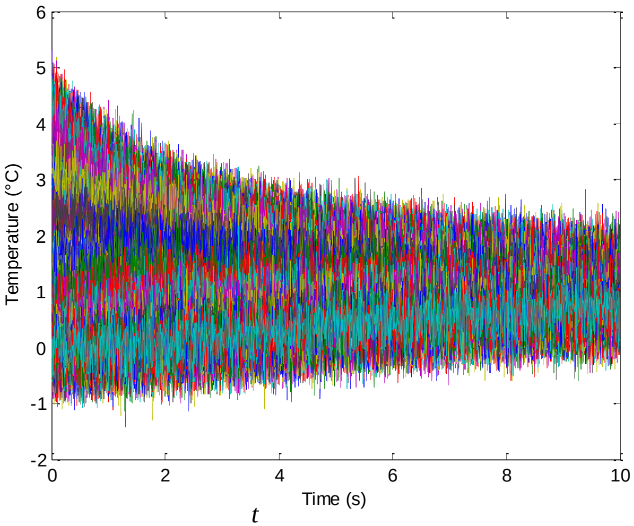
Noise ampl. $(°C)$	$\hat{\beta}$ (s^{-1})	$\left\lvert\frac{\beta-\hat{\beta}}{\beta}\right\rvert\times100$	$\hat{\alpha}_x$ $(\times10^{-6} m^2 s^{-1})$	$\left\lvert\frac{\alpha_x-\hat{\alpha}_x}{\alpha_x}\right\rvert\times100$	$\hat{\alpha}_y$ $(\times10^{-6} m^2 s^{-1})$	$\left\lvert\frac{\alpha_y-\hat{\alpha}_y}{\alpha_y}\right\rvert\times100$
± 1.00	0.0152 $(\pm0.5\text{x}10^{-3})$	0.055	0.1572 (± 0.02)	3.72	0.2889 (±0.036)	4.66
± 0.50	0.0152 $(\pm0.2\text{x}10^{-3})$	0.06	0.1540 (± 0.007)	1.62	0.2989 (±0.0162)	1.37
± 0.10	0.0152 $(\pm0.04\text{x}10^{-3})$	0.002	0.1517 (± 0.0015)	0.12	0.3026 (±0.0037)	0.15
± 0.02	0.0152 $(\pm0.01\text{x}10^{-3})$	0.0007	0.1515 (± 0.0003)	0.022	0.3029 (±0.0007)	0.0003

Estimation improvement required

ESTIMATIONS IMPROVEMENT
Cumulative integral-SVD method

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha_x \frac{\partial^2 T(x,y,t)}{\partial x^2} + \alpha_y \frac{\partial^2 T(x,y,t)}{\partial y^2} - \beta T(x,y,t)$$

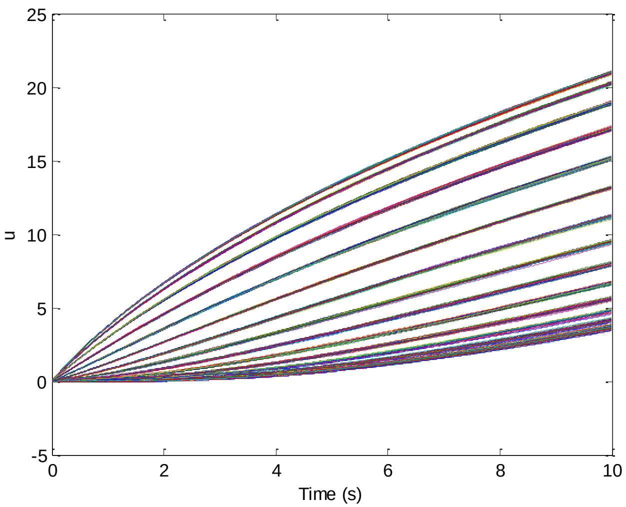
+ adiabatic boundary conditions



$$u(x,y,t) = \int_0^t T(x,y,\tau)d\tau$$

$$\Delta T(x,y,t) = \alpha_x \frac{\partial^2 u(x,y,t)}{\partial x^2} + \alpha_y \frac{\partial^2 u(x,y,t)}{\partial y^2} - \beta u(x,y,t)$$

+ adiabatic boundary conditions & $\Delta T(x,y,t) = T(x,y,t) - T_o(x,y)$



SVD: $u(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) z_m(t)$

Projection: $\Delta T(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) \eta_m(t)$

$$\boxed{\begin{bmatrix} \hat{\alpha}_x \\ \hat{\alpha}_y \end{bmatrix} = \mathbf{M}^{-1}\mathbf{y}} \qquad \mathbf{y} = \begin{bmatrix} \left\langle \tilde{z}_1 \tilde{\eta}_1 \right\rangle_t / \tilde{\sigma}_1^2 + \hat{\beta} \\ \left\langle \tilde{z}_1 \tilde{\eta}_2 \right\rangle_t / \tilde{\sigma}_1^2 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \left\langle \partial_x^2 V_1(x,y), V_1(x,y) \right\rangle_\Omega & \left\langle \partial_y^2 V_1(x,y), V_1(x,y) \right\rangle_\Omega \\ \left\langle \partial_x^2 V_1(x,y), V_2(x,y) \right\rangle_\Omega & \left\langle \partial_y^2 V_1(x,y), V_2(x,y) \right\rangle_\Omega \end{bmatrix}$$

ESTIMATIONS IMPROVEMENT
Modal-SVD method

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha_x \frac{\partial^2 T(x,y,t)}{\partial x^2} + \alpha_y \frac{\partial^2 T(x,y,t)}{\partial y^2} - \beta T(x,y,t)$$

+ adiabatic boundary conditions

$$\left\{ \begin{array}{l} T(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) z_m(t) \\ \left\langle \circ, z_j(t) \right\rangle_t \\ \left\langle z_m(t), z_j(t) \right\rangle_t \equiv \int_t z_m(t) z_j(t) dt = \delta_{mj} \sigma_j^2 \end{array} \right.$$

Set of coupled PDE
for eigenfunctions

$$\alpha_x \frac{\partial^2 V_j(x,y)}{\partial x^2} + \alpha_y \frac{\partial^2 V_j(x,y)}{\partial y^2} - \beta V_j(x,y) = \sum_{m=1}^{\infty} V_m(x,y) a_{mj}$$

+ adiabatic boundary conditions

with: $\forall (m,j) \quad a_{mj} = a_{jm} = \dfrac{z_m(t_f) z_j(t_f) - z_m(0) z_j(0)}{\sigma_m^2 + \sigma_j^2}$

$$\alpha_x \begin{bmatrix} \partial_x^2 V_1(x,y) \\ \partial_x^2 V_2(x,y) \\ \vdots \end{bmatrix} + \alpha_y \begin{bmatrix} \partial_y^2 V_1(x,y) \\ \partial_y^2 V_2(x,y) \\ \vdots \end{bmatrix} - \beta \begin{bmatrix} V_1(x,y) \\ V_2(x,y) \\ \vdots \end{bmatrix} = \mathbf{A} \begin{bmatrix} V_1(x,y) \\ V_2(x,y) \\ \vdots \end{bmatrix}$$

$$\mathbf{A} = \{a_{mi}\}$$

$$a_{mi} = \frac{z_m(t_f)z_i(t_f) - z_m(0)z_i(0)}{\sigma_m^2 + \sigma_i^2}$$

$$\begin{bmatrix} \phi_1(x,y) \\ \phi_2(x,y) \\ \vdots \end{bmatrix} = \mathbf{P}^t \begin{bmatrix} V_1(x,y) \\ V_2(x,y) \\ \vdots \end{bmatrix}$$

$$\mathbf{A} = \mathbf{P}\,\mathbf{\Sigma}\mathbf{P}^t$$

$$\mathbf{P}^t\mathbf{P} = \mathbf{I}; \quad \mathbf{\Sigma} = diag\{\lambda_j\}$$

Set of independent PDE

$$\alpha_x \frac{\partial^2 \phi_j(x,y)}{\partial x^2} + \alpha_y \frac{\partial^2 \phi_j(x,y)}{\partial y^2} - \beta\phi_j(x,y) = \lambda_j \phi_j(x,y)$$

+ adiabatic boundary conditions

Eigenfunctions/Eigenvalues problem associated to the thermal equations

Set of independent PDE

$$\alpha_x \frac{\partial^2 \phi_j(x,y)}{\partial x^2} + \alpha_y \frac{\partial^2 \phi_j(x,y)}{\partial y^2} - \beta \phi_j(x,y) = \lambda_j \phi_j(x,y)$$

+ adiabatic boundary conditions

Eigenfunctions/Eigenvalues problem associated to the thermal equations

Orthomormal $\phi_j(x,y)$

$$\begin{cases} \alpha_x \left\langle \partial_x^2 \phi_j(x,y), \phi_j(x,y) \right\rangle_\Omega + \alpha_y \left\langle \partial_x^2 \phi_j(x,y), \phi_j(x,y) \right\rangle = \lambda_j + \beta \\ \alpha_x \left\langle \partial_x^2 \phi_j(x,y), \phi_k(x,y) \right\rangle_\Omega + \alpha_y \left\langle \partial_x^2 \phi_j(x,y), \phi_k(x,y) \right\rangle = 0 \end{cases}$$

Analytical

$$\begin{cases} \phi_j(x,y) = (2/L)\cos(\kappa_p x)\cos(\gamma_s y) \\ \kappa_p = p\pi/L;\ \gamma_s = s\pi/L;\ \left\langle \partial_x^2 \phi_j, \phi_j \right\rangle = -\kappa_p^2;\ \left\langle \partial_y^2 \phi_j, \phi_j \right\rangle = -\gamma_s^2 \end{cases}$$

$$\boxed{\alpha_x \kappa_{pj}^2 + \alpha_y \gamma_{sj}^2 = -(\lambda_j + \beta)}$$

IN PRACTICE - Without noise

$$\alpha_x \frac{\partial^2 V_j(x,y)}{\partial x^2} + \alpha_y \frac{\partial^2 V_j(x,y)}{\partial y^2} - \beta V_j(x,y) = \sum_{m=1}^{\infty} V_m(x,y) a_{mj}$$

$$[\alpha_x \mathbf{L}_x + \alpha_y \mathbf{L}_y - \beta\, \mathbf{I}]\, \mathbf{V} = \mathbf{VA}$$

$$\mathbf{A} = \mathbf{P}\, \boldsymbol{\Sigma} \mathbf{P}^t$$

$$\boldsymbol{\varphi} = \mathbf{VP}$$

$$\alpha_x \underbrace{\boldsymbol{\varphi}^t \mathbf{L}_x \boldsymbol{\varphi}}_{\text{diag}} + \alpha_y \underbrace{\boldsymbol{\varphi}^t \mathbf{L}_y \boldsymbol{\varphi}}_{\text{diag}} = \boldsymbol{\Sigma} + \beta\, \mathbf{I}$$

n equations

$$\alpha_x \kappa_{pj}^2 + \alpha_y \gamma_{sj}^2 = -(\lambda_j + \beta)$$

$$[\alpha_x \mathbf{L}_x + \alpha_y \mathbf{L}_y - \beta\, \mathbf{I}] \begin{bmatrix} \mathbf{V}_d & \mathbf{V}_f \end{bmatrix} = \begin{bmatrix} \mathbf{V}_d & \mathbf{V}_f \end{bmatrix} \begin{bmatrix} \mathbf{A}_{dd} & \mathbf{A}_{fd} \\ \mathbf{A}_{fd} & \mathbf{A}_{fd} \end{bmatrix}$$

$$[\alpha_x \mathbf{L}_x + \alpha_y \mathbf{L}_y - \beta\, \mathbf{I}]\, \mathbf{V}_d = \mathbf{V}_d \mathbf{A}_{dd} + \mathbf{V}_f \mathbf{A}_{fd}$$

$$\mathbf{V}_d^t \mathbf{V}_f = \mathbf{0}$$

$$\mathbf{V}_d^t [\alpha_x \mathbf{L}_x + \alpha_y \mathbf{L}_y - \beta\, \mathbf{I}]\, \mathbf{V}_d = \mathbf{A}_{dd}$$

$$\mathbf{A}_{dd} = \mathbf{P}_d\, \boldsymbol{\Sigma}_d \mathbf{P}_d^t$$

$$\boldsymbol{\varphi}_d = \mathbf{V}_d \mathbf{P}_d$$

$$\alpha_x \underbrace{\boldsymbol{\varphi}_d^t \mathbf{L}_x \boldsymbol{\varphi}_d}_{\text{non diag}} + \alpha_y \underbrace{\boldsymbol{\varphi}_d^t \mathbf{L}_y \boldsymbol{\varphi}_d}_{\text{non diag}} = \boldsymbol{\Sigma}_d + \beta\, \mathbf{I}_d$$

MAIS

$$\boxed{\begin{array}{c} \boldsymbol{\Sigma}_d \overset{\sim}{\subset} \boldsymbol{\Sigma} \\ eig(\boldsymbol{\varphi}_d^t \mathbf{L}_{x,y} \boldsymbol{\varphi}_d) \overset{\sim}{\subset} eig(\boldsymbol{\varphi}^t \mathbf{L}_{x,y} \boldsymbol{\varphi}) \end{array}}$$

d equations

$$\alpha_x \kappa_{pj}^2 + \alpha_y \gamma_{sj}^2 = -(\lambda_j + \beta)$$

IN PRACTICE - With noise

$$\tilde{\mathbf{T}}(t) \Rightarrow \mathbf{V}_d, \tilde{\sigma}_1^2, \tilde{\sigma}_2^2, \cdots, \tilde{\sigma}_d^2, \tilde{z}_1(t), \tilde{z}_2(t), \cdots, \tilde{z}_d(t)$$

$$\tilde{a}_{mi} = \frac{\tilde{z}_m(t_f)\tilde{z}_i(t_f) - \tilde{z}_m(0)\tilde{z}_i(0)}{\tilde{\sigma}_m^2 + \tilde{\sigma}_i^2};\ m,i = 1,2 \Rightarrow \tilde{\mathbf{A}}_{dd} \quad (d \times d)$$

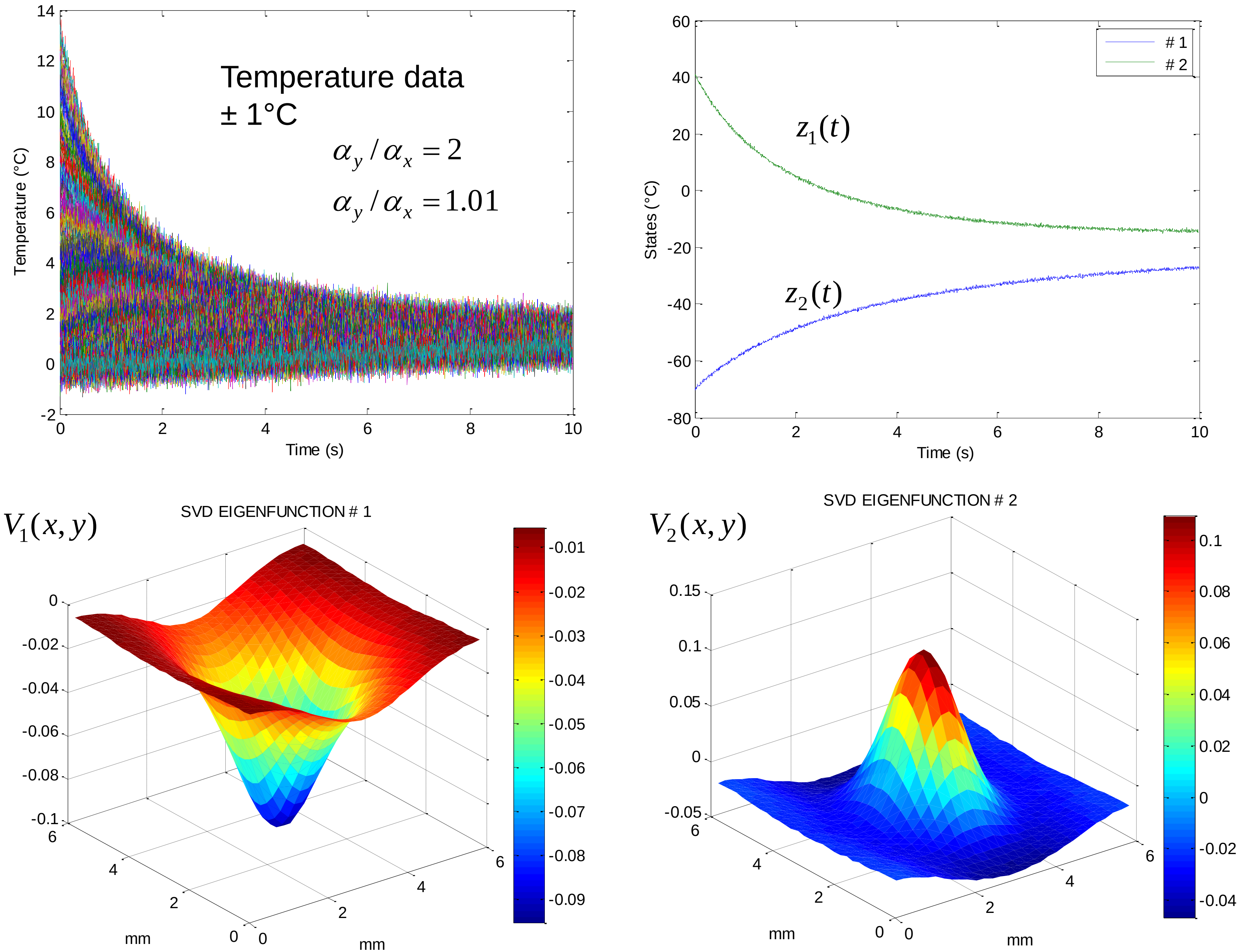
$$\tilde{\mathbf{A}}_{dd} = \tilde{\mathbf{P}}_d\, \tilde{\boldsymbol{\Sigma}}_d \tilde{\mathbf{P}}_d^t$$

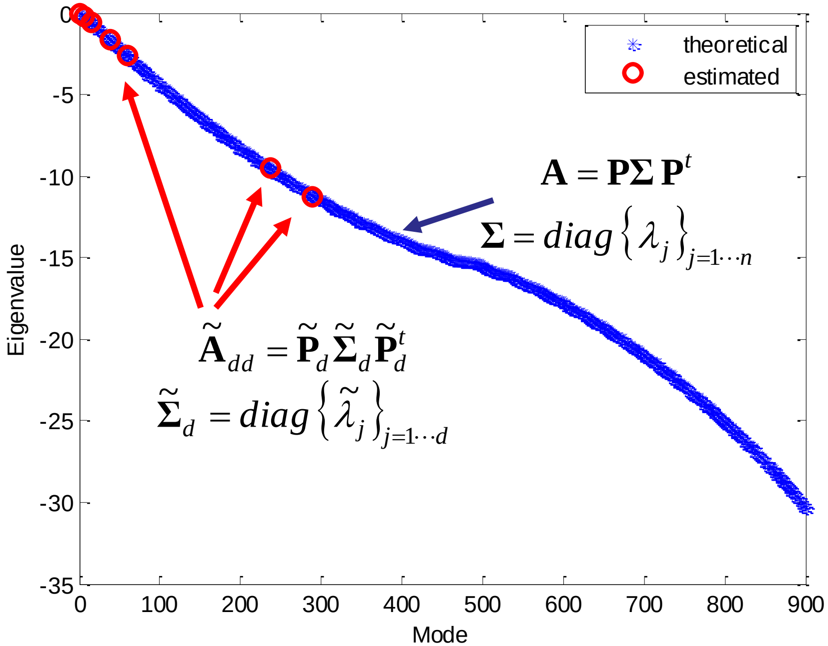
$$\tilde{\boldsymbol{\varphi}}_d = \tilde{\mathbf{V}}_d \tilde{\mathbf{P}}_d \xrightarrow{\text{Closest analytical OX \& OY eigenvalues}} \begin{array}{l} \kappa_{p1}, \kappa_{p2}, \cdots, \kappa_{pd} \\ \gamma_{s1}, \gamma_{s2}, \cdots, \gamma_{sd} \end{array}$$

$$eig(\boldsymbol{\varphi}_d^t \mathbf{L}_x \boldsymbol{\varphi}_d) \rightarrow \tilde{\kappa}_{p1}, \tilde{\kappa}_{p2}, \cdots, \tilde{\kappa}_{d1}$$

$$eig(\tilde{\boldsymbol{\varphi}}_d^t \mathbf{L}_y \tilde{\boldsymbol{\varphi}}_d) \rightarrow \tilde{\gamma}_{s1}, \tilde{\gamma}_{s2}, \cdots, \tilde{\gamma}_{sd}$$

$$\left.\begin{array}{l} \alpha_x \kappa_{p1}^2 + \alpha_y \gamma_{s1}^2 = -(\tilde{\lambda}_1 + \hat{\beta}) \\ \alpha_x \kappa_{p2}^2 + \alpha_y \gamma_{s2}^2 = -(\tilde{\lambda}_2 + \hat{\beta}) \\ \cdots\cdots \\ \alpha_x \kappa_{pd}^2 + \alpha_y \gamma_{sd}^2 = -(\tilde{\lambda}_d + \hat{\beta}) \end{array}\right\} \begin{array}{l} \hat{\alpha}_x \\ \hat{\alpha}_y \end{array}$$


Temperature data
± 1°C
$\alpha_y / \alpha_x = 2$
$\alpha_y / \alpha_x = 1.01$
Temperature (°C)
Time (s)
1
2
$z_1(t)$
$z_2(t)$
States (°C)
Time (s)
$V_1(x,y)$
SVD EIGENFUNCTION # 1
mm
mm
$V_2(x,y)$
SVD EIGENFUNCTION # 2
mm
mm



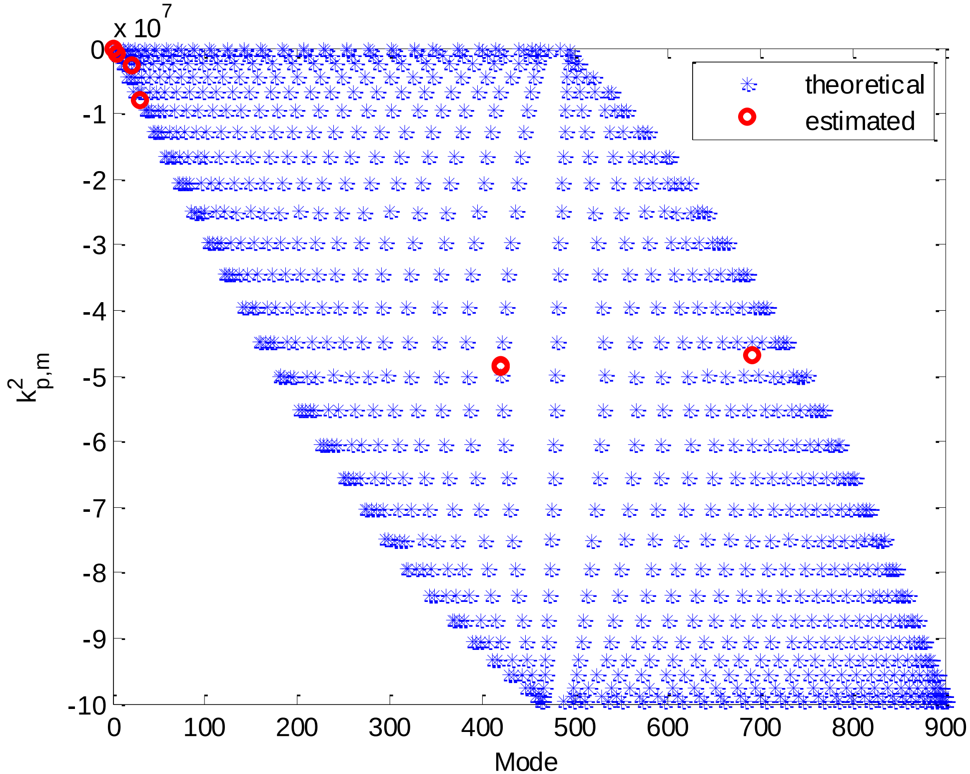
λ_j	$\tilde{\lambda}_j$
-0.0152	-0.0178
-0.1824	-0.1914
-0.5552	-0.5856
-1.6420	-1.6483
-2.5883	-2.5882
-9.5405	-9.5357
-11.2034	-11.2358

$\boldsymbol{\varphi}_d^t \mathbf{L}_x \boldsymbol{\varphi}_d$ non diagonal

-0.0041	0.0040	0.0074	0.0108	0.0307	0.0125	0.0002
0.0040	-0.0813	-0.0284	-0.0014	-0.0814	0.0102	-0.0241
0.0074	-0.0284	-0.3358	0.1455	-0.1605	0.2043	-0.1677
0.0108	-0.0014	0.1455	-1.8630	-1.6206	-0.7192	0.3088
0.0307	-0.0814	-0.1605	-1.6206	-4.0645	0.3012	-0.4070
0.0125	0.0102	0.2043	-0.7192	0.3012	-4.5169	-0.2379
0.0002	-0.0241	-0.1677	0.3088	-0.4070	-0.2379	-4.8431

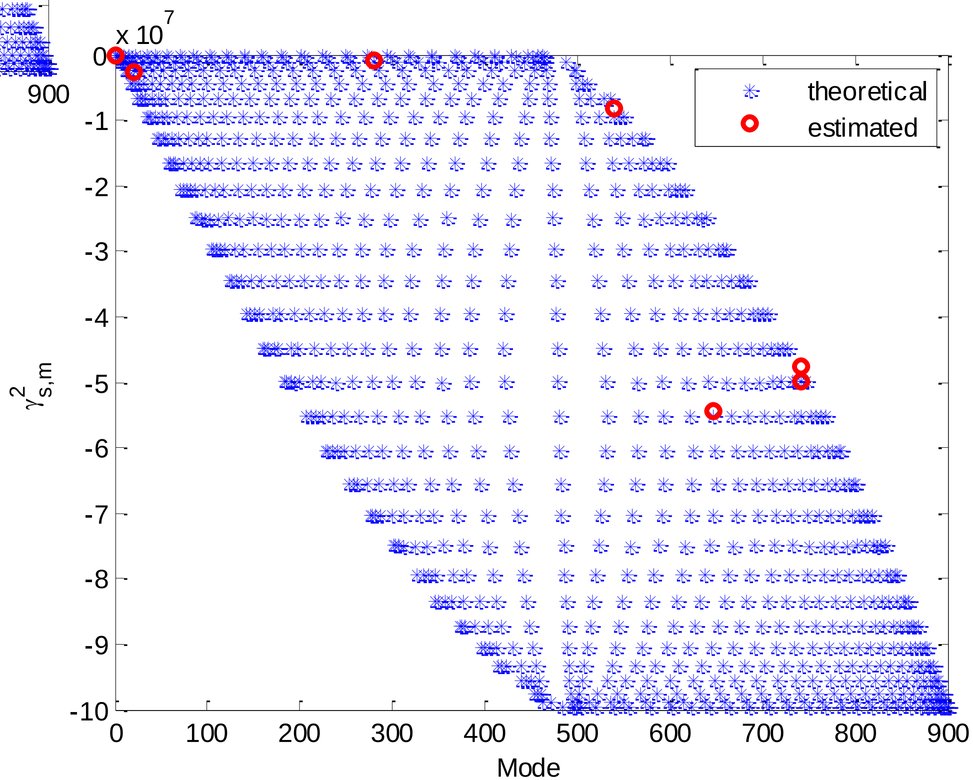
k_j^2	$\tilde{k}_j^2$
-0.6699	-0.7892
-0.2447	-0.2697
-0.0000	-0.0036
-0.1093	-0.0771
-4.4774	-4.6858
-5.0000	-4.8604
-5.0000	-4.8391

$eig(\boldsymbol{\varphi}_d^t \mathbf{L}_x \boldsymbol{\varphi}_d)$



$$\tilde{\kappa}_{p,m} = \tilde{p}\pi / L \Rightarrow \tilde{p} \Rightarrow p \Rightarrow \kappa_{p,m} = p\pi / L$$

$$\tilde{\gamma}_{s,m} = \tilde{s}\pi / L \Rightarrow \tilde{s} \Rightarrow s \Rightarrow \gamma_{s,m} = s\pi / L$$



$$\alpha_y / \alpha_x = 2$$

Noise amplitude ± 1.00 °C	$\hat{\alpha}_x$ $(\times 10^{-6} m^2 s^{-1})$	$\left\lvert \frac{\alpha_x - \hat{\alpha}_x}{\alpha_x} \right\rvert \times 100$	$\hat{\alpha}_y$ $(\times 10^{-6} m^2 s^{-1})$	$\left\lvert \frac{\alpha_y - \hat{\alpha}_y}{\alpha_y} \right\rvert \times 100$
SVD	0.1468 (± 0.018)	3.13	0.3091 (±0.028)	2.00
CI-SVD	0.1468 (± 0.015)	3.12	0.3091 (±0.025)	2.00
Modal-SVD	**0.1515 (± 0.0012)**	**0.009**	**0.3030 (±0.0026)**	**0.0013**

Noise amplitude ± 1.00 °C	$\hat{\alpha}_x$ $(\times 10^{-6} m^2 s^{-1})$	$\left\lvert \frac{\alpha_x - \hat{\alpha}_x}{\alpha_x} \right\rvert \times 100$	$\hat{\alpha}_y$ $(\times 10^{-6} m^2 s^{-1})$	$\left\lvert \frac{\alpha_y - \hat{\alpha}_y}{\alpha_y} \right\rvert \times 100$
SVD	0.0286 (± 1.2)	81.12	0.2760 (±1.0)	80.37
CI-SVD	- 0.0036 (± 1.0)	102.3	0.3085 (±1.0)	101.6
Modal-SVD	**0.1524 (± 0.0014)**	**0.58**	**0.1539 (±0.0013)**	**0.585**

$$\alpha_y / \alpha_x = 1.01$$

ESTIMATIONS IMPROVEMENT
Modal approach

$$\phi_j(x,y) = (2/L)\cos(\kappa_p x)\cos(\gamma_s y)$$
$$\kappa_p = p\pi/L;\ \gamma_s = s\pi/L$$

$$\boxed{\alpha_x \kappa_{pj}^2 + \alpha_y \gamma_{sj}^2 = -(\lambda_j + \beta)}$$

$$T(x,y,t) = \sum_{j=1}^{\infty} \phi_j(x,y)\eta_j(t)$$
$$\eta_j(t) = \eta_j(0)\exp(-\lambda_j t)$$

$$\tilde{\boldsymbol{\eta}}(t) = \boldsymbol{\varphi}^t \tilde{\mathbf{T}}(t)$$

$$diag\left[\int_{t=0}^{t_{fin}} \tilde{\boldsymbol{\eta}}(t)\tilde{\boldsymbol{\eta}}^t(t)dt\right]$$

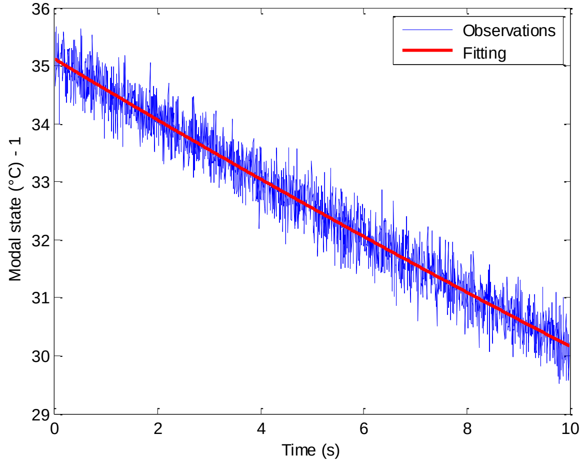
$$\boxed{\alpha_x \kappa_{pj}^2 + \alpha_y \gamma_{sj}^2 = -(\tilde{\lambda}_j + \hat{\beta})}$$

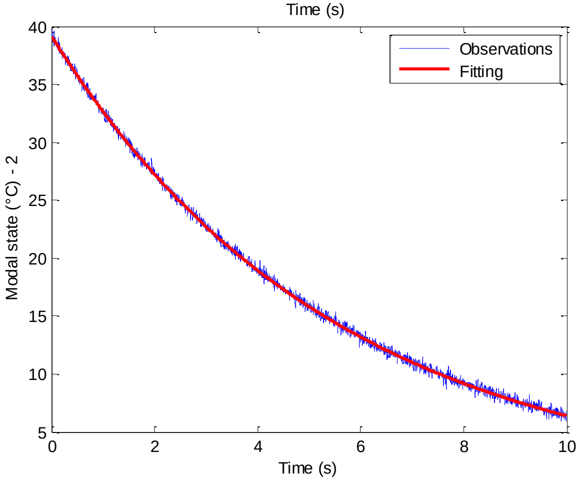
Selection of 3 states
Those with greater energy

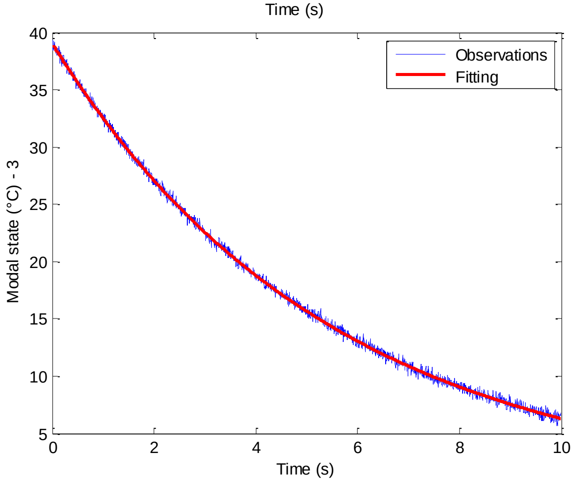
$$\tilde{\eta}_j(t) \quad j = a,b,c \Rightarrow \kappa_{pj}^2,\ \gamma_{sj}^2$$

Eigenvalues estimation
(minimum least squares)

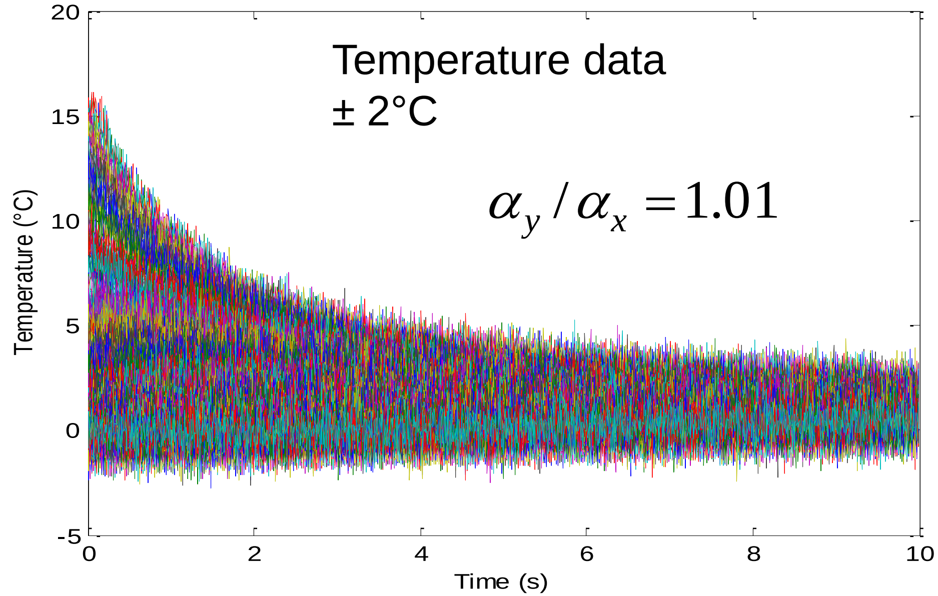
$$\eta_j(t) = \eta_j(0)\exp(-\lambda_j t)$$







λ_j	$\tilde{\lambda}_j$	p	s
-0.0152	-0.0151	0	0
-0.1807	-0.1808	2	0
-0.1824	-0.1827	0	2



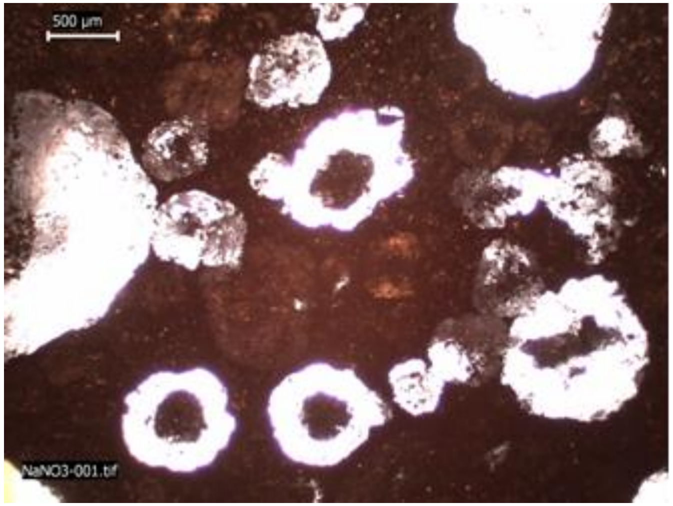
$$\hat{\alpha}_x = 0.1518 \pm 0.0009 mm^2/s \ \ (bias: 0.20\%)$$
$$\hat{\alpha}_y = 0.1534 \pm 0.0008 mm^2/s \ \ (bias: 0.26\%)$$

MODAL

$$\hat{\alpha}_x = 0.1517 \pm 0.0105 mm^2/s \ \ (bias: 0.13\%)$$
$$\hat{\alpha}_y = 0.1535 \pm 0.0029 mm^2/s \ \ (bias: 0.32\%)$$

MODAL SVD

Heterogeneous materials



Adiabatic boundary conditions

p non overlapping homogeneous phases paving the domain

Within the phases

$$\frac{\partial T(\mathbf{x},t)}{\partial t} = \alpha_i \nabla^2 T(\mathbf{x},t) - \beta_i T(\mathbf{x},t)$$

At the interfaces

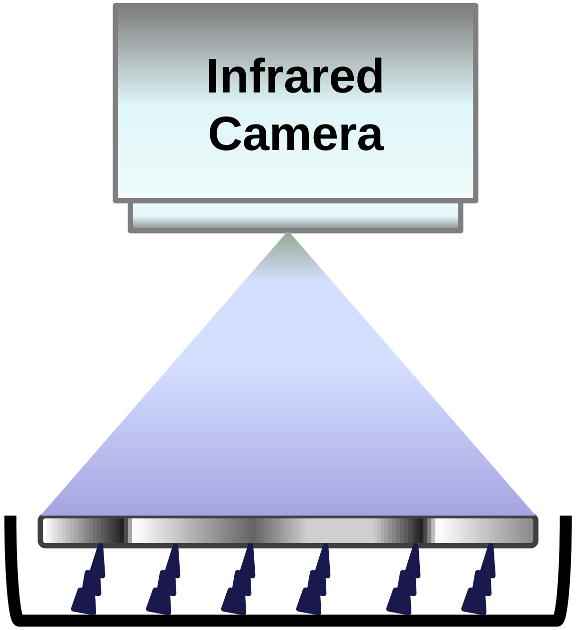
$$-k_i \nabla T(\mathbf{x},t)\big|^{i} \mathbf{n}_{ij} = -h_{ij}\left(T(\mathbf{x},t)\big|^{i} - T(\mathbf{x},t)\big|^{j}\right)$$

$$-k_j \nabla T(\mathbf{x},t)\big|^{j} \mathbf{n}_{ij} = -h_{ij}\left(T(\mathbf{x},t)\big|^{j} - T(\mathbf{x},t)\big|^{i}\right)$$

Microstructure
Phases discrimination
Interfaces location

→ ***Thermal diffusivity of the phases*** → ***Thermal resistance at the interfaces***

MICROSTRUCTURE RETRIEVAL
Phases discrimination – Interfaces location



Applying a uniform heat flux on the back side of the sample during a very short time & recording the thermal behavior of the plate via an IR camera

Negligible/limited heat diffusion between adjacent pixels

The thermal behavior of a pixel is determined by its thermal capacity

All the pixels belonging to the same phase behave in the same way

$$\forall (x, y) \in \Omega_i$$
$$T_i(t) = (\varphi / \beta_i)(1 - \exp(-\beta_i t))$$

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha_i \nabla^2 T(x,y,t) - \beta_i T(x,y,t) + \varphi$$

+ Robin conditions at the interfaces
+ Adiabatic boundary conditions

$$W(\mathbf{x},\mathbf{x}') = \sum_{m=1}^{\infty} \sigma_m^2 V_m(\mathbf{x}) V_m(\mathbf{x}')$$

$$\forall t,\ T(\mathbf{x},t) = \sum_{m=1}^{\infty} V_m(\mathbf{x}) z_m(t)$$

SVD – Noise, parsimony

Likeness

IDEAL CASE
With infinite thermal
resistance at the interfaces

$$\forall (x,y) \in \Omega_i$$
$$T_i(t) = (\varphi / \beta_i)(1 - \exp(-\beta_i t))$$

Number of phases

Rank of the energy matrix W =
Number of phases

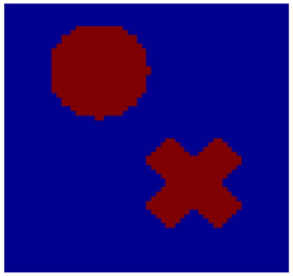
Discontinuity of the eigenfunctions at the interfaces

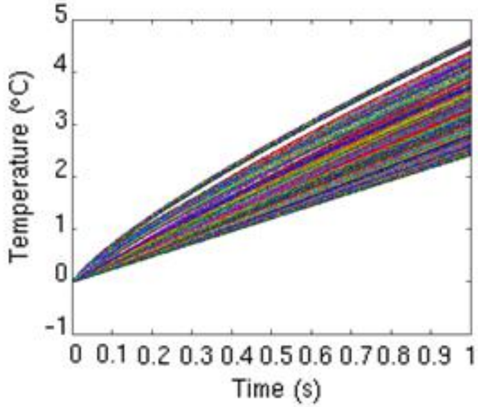
$$\forall (x,y)\ \&\ (x',y') \in \Omega_i, \forall m \Rightarrow V_m(x,y) = V_m(x',y')$$
$$\forall (x,y) \in \Omega_i\ \&\ (x',y') \in \Omega_j, \forall m \Rightarrow V_m(x,y) \neq V_m(x',y')$$

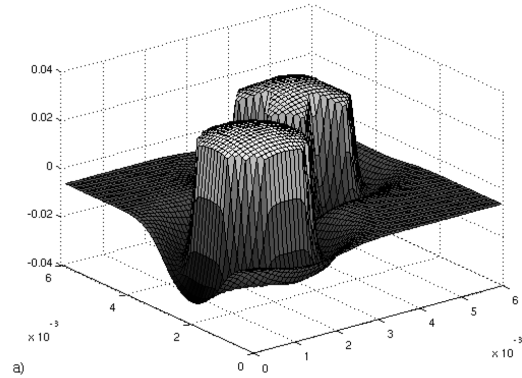
Eigenfunctions sign
→ Criteria for phases discrimination

$$\forall m > 1\ \ \langle V_m(x,y) \rangle_{\Omega} = 0$$
$$\forall (x,y) \in \Omega_i\ \&\ (x',y') \in \Omega_j, \exists m / V_m(x,y) V_m(x',y') < 0$$

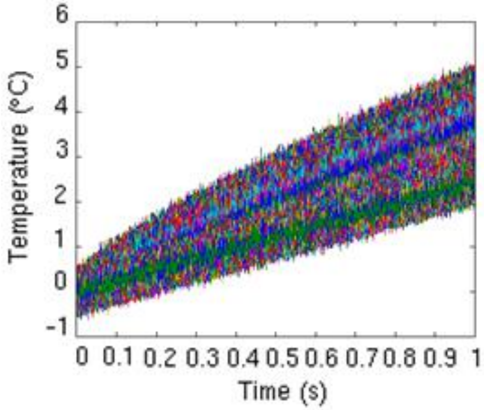
NUMERICAL TEST

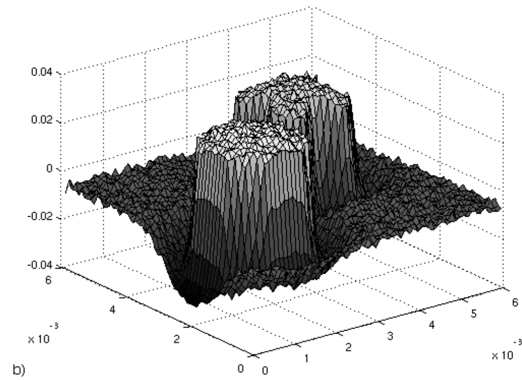




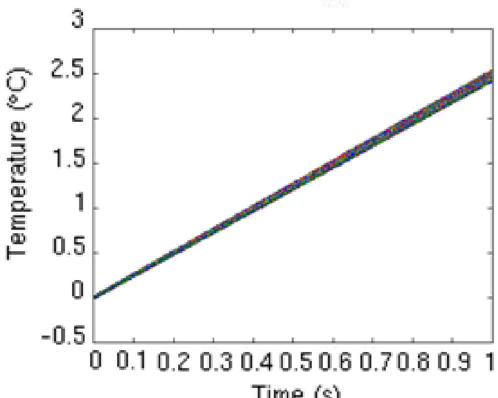


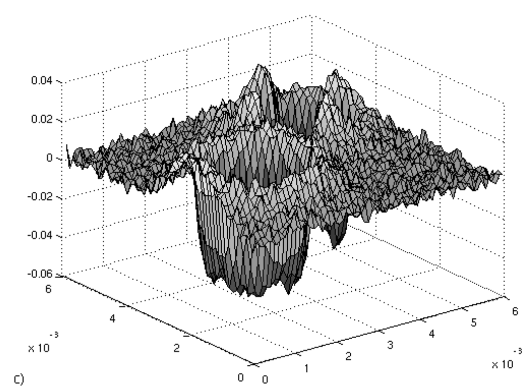
High contrast between phases in terms of thermal properties $\left(\alpha_b/\alpha_r = 134.7\right)$
High quality measurements (±0.02°C)



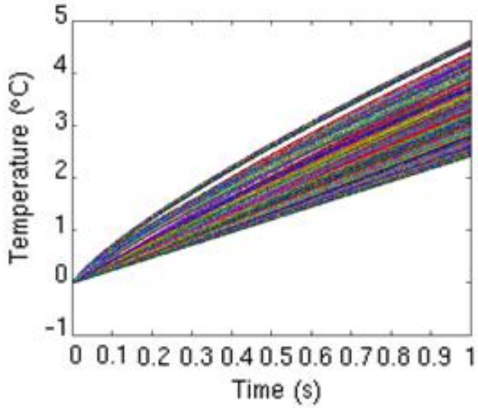


High contrast between phases in terms of thermal properties $\left(\alpha_b/\alpha_r = 134.7\right)$
Bad quality measurements (±0.5°C)

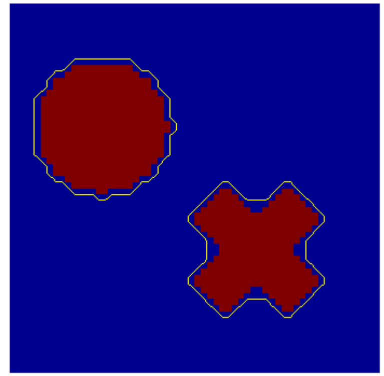




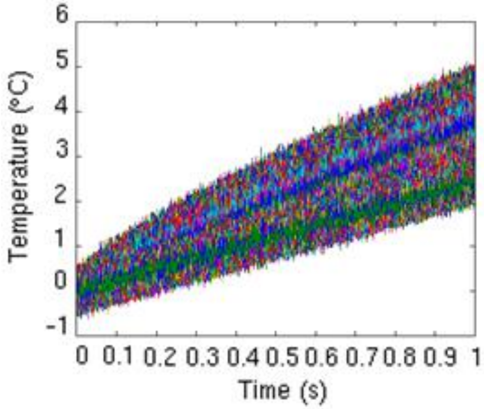
Very low contrast between phases in terms of thermal properties $\left(\alpha_b/\alpha_r = 1.05\right)$
High quality measurements (±0.02°C)



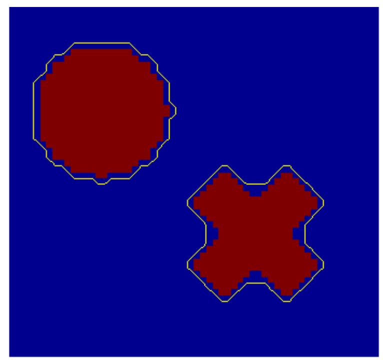
a)



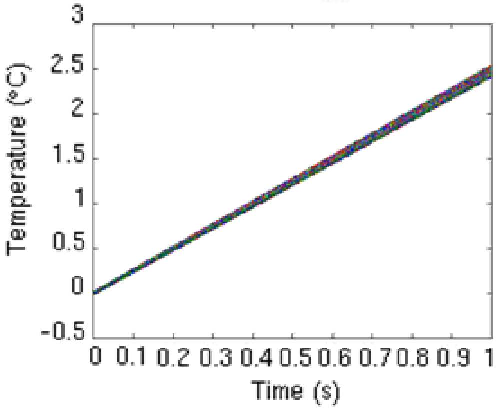
High contrast between phases in terms of thermal properties $\left(\alpha_b/\alpha_r = 134.7\right)$
High quality measurements (±0.02°C)



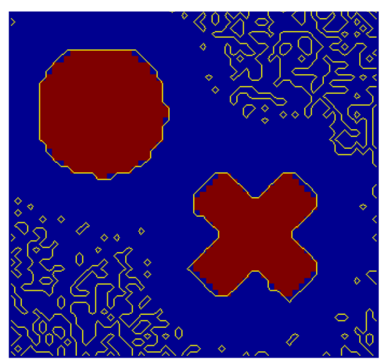
b)



High contrast between phases in terms of thermal properties $\left(\alpha_b/\alpha_r = 134.7\right)$
Bad quality measurements (±0.5°C)

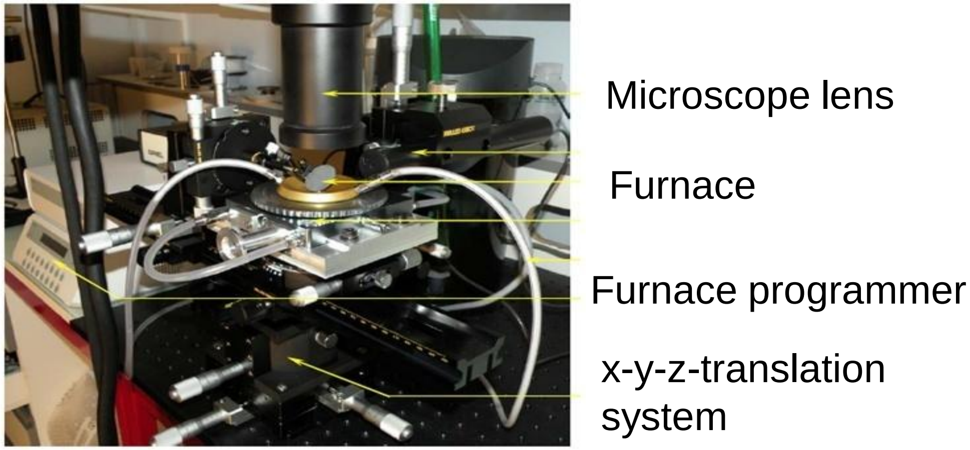


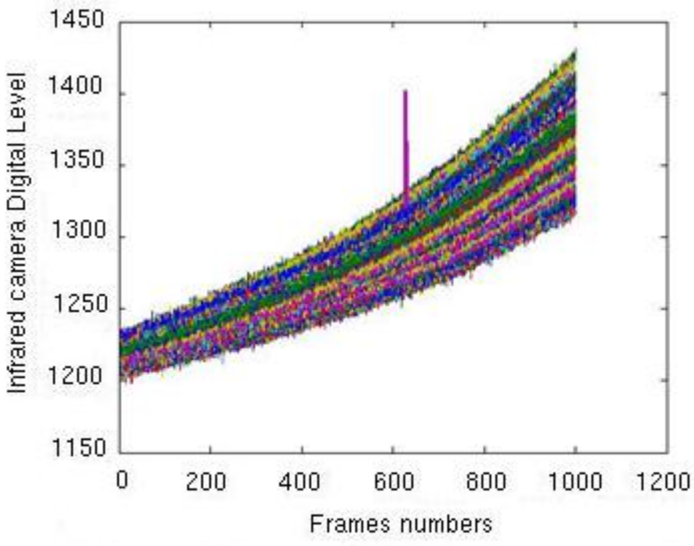
c)



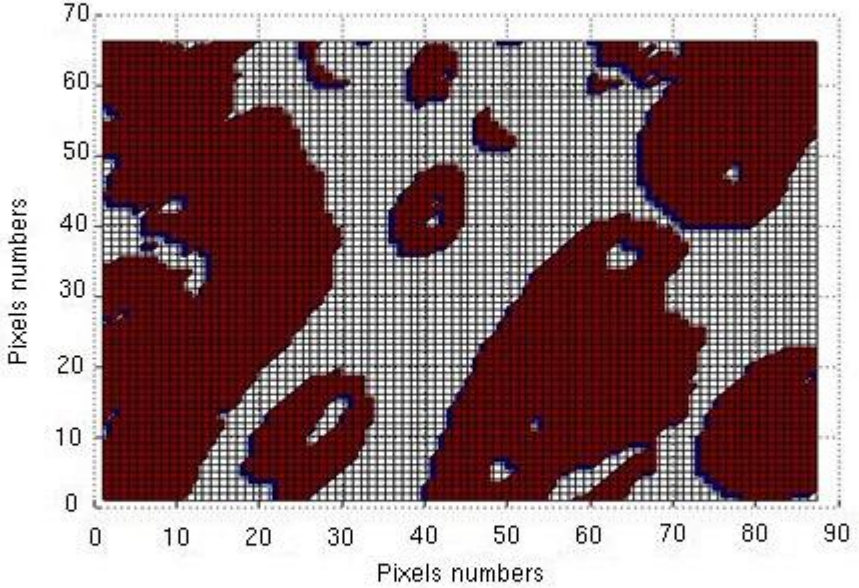
Very low contrast between phases in terms of thermal properties $\left(\alpha_b/\alpha_r = 1.05\right)$
High quality measurements (±0.02°C)

EXPERIMENTAL TEST

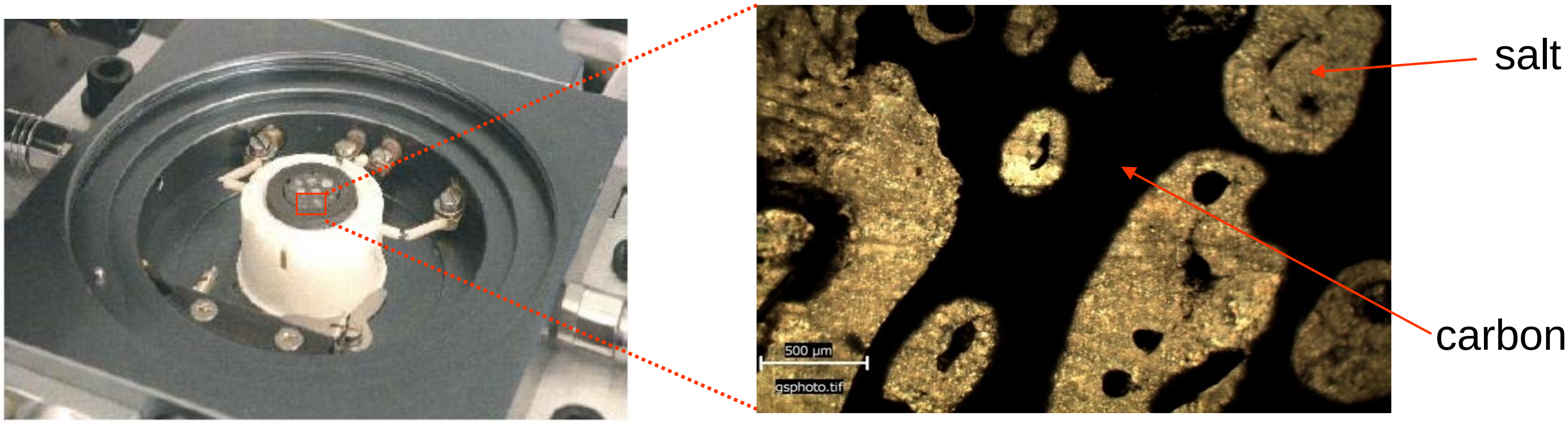




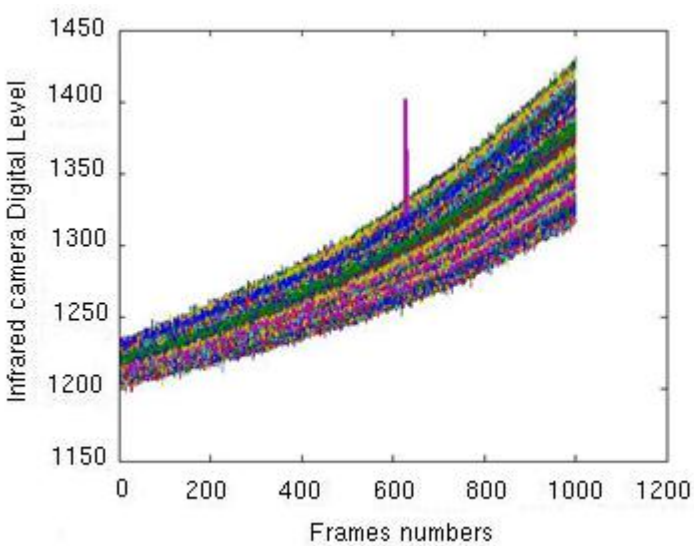
Thermal response



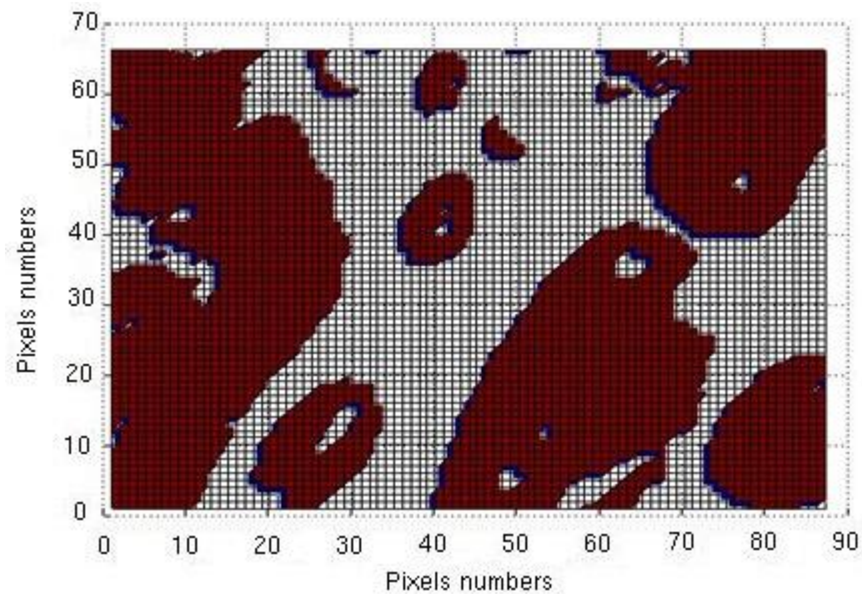
$\mathrm{sign}(V_2(x, y))$



Optical microscopic picture



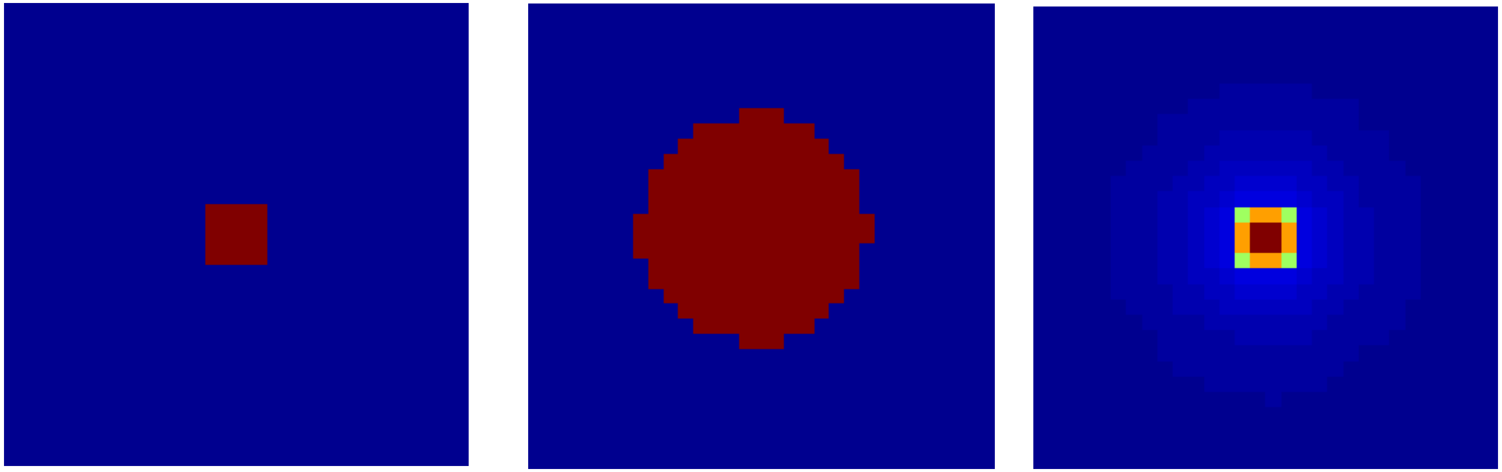
Thermal response



$\mathrm{sign}(V_2(x, y))$

… but

$\mathrm{sgn}(V_2(x,y))$ $\quad$ $V_2(x,y)$



$$u(x,y,t) = \int_0^t T(x,y,\tau)\,d\tau$$

$$u(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) z_m(t)$$

$$\Omega_i : f_m(x,y) = \alpha_i \nabla^2 V_m(x,y,t) - \beta_i V_m(x,y,t)$$

$$\Gamma : \begin{cases} k_i \partial_n V_m(x,y,t)\big|_i - k_j \partial_n V_m(x,y,t)\big|_j = 0 \\ V_m(x,y,t)\big|_i = V_m(x,y,t)\big|_j \end{cases}$$

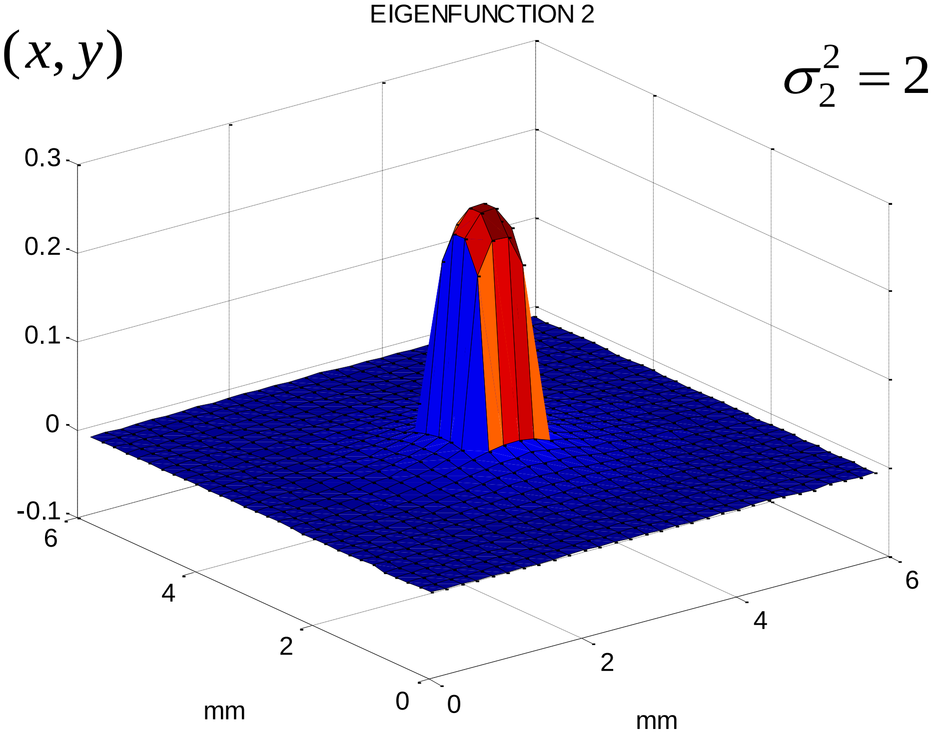
with $f_m(x,y) = (1/\sigma_m^2)\left[\langle T, z_m\rangle_t - \varphi \langle t, z_m\rangle_t\right]$

Level-set method

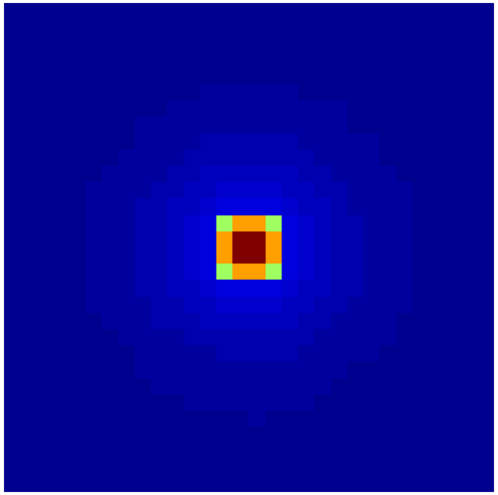
Look for Γ that minimizes the cost function

$$J = \int_\Omega (\tilde{f}_m(x,y) - f_m(x,y))^2\,dxdy$$

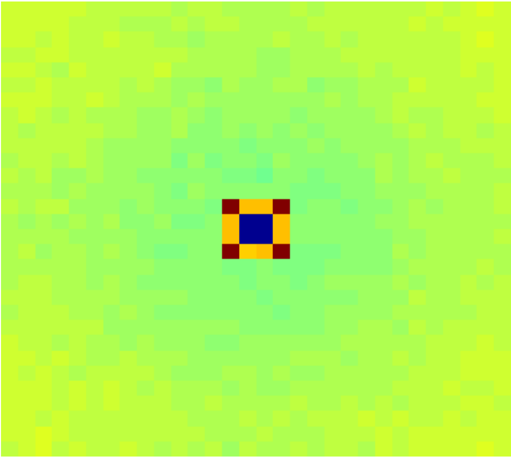
$V_2(x, y)$



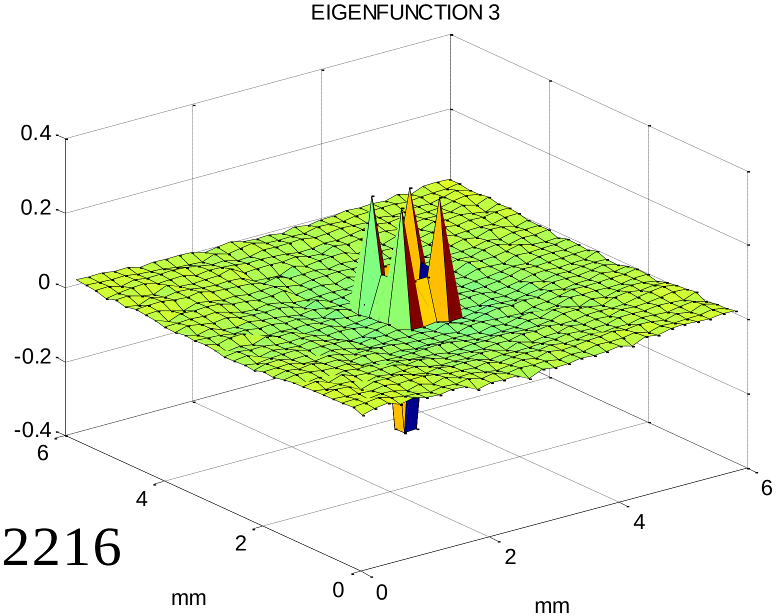
$\sigma_2^2 = 230.74$



$V_3(x, y)$



$\sigma_3^2 = 2.2216$



DIFFUSIVITIES ESTIMATION
SVD-based method

$$\frac{\partial T(x,y,t)}{\partial t} = \alpha_i \nabla^2 T(x,y,t) - \beta_i T(x,y,t)$$

+ Robin & adiabatic boundary conditions

$$T(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) z_m(t)$$

$$\langle V_k, V_m \rangle_\Omega \equiv \int_\Omega V_k(x) V_m(x) dx = \delta_{km}$$

$$\langle z_m(t), z_k(t) \rangle_t \equiv \int_t z_m(t) z_k(t) dt = \delta_{mk} \sigma_m^2$$

Integration over Ω

$$\sum_{i=1}^{p} \frac{\varepsilon_i}{\beta_i} \frac{d\langle T \rangle_{\Omega_i}}{dt} = -\langle T \rangle_\Omega$$

ESTIMATION β1 & β2

linear least squares methods

$$\langle \langle \circ, V_k \rangle_\Omega, z_k(t) \rangle_t$$

$$i = 1, 2, 3, \cdots; \quad k = i, i+1, i+2, \cdots$$

$$y_k = \sum_{i=1}^{p} \alpha_i \langle \nabla^2 V_k(x,y), V_k(x,y) \rangle_{\Omega_i} - \sum_{i=1}^{p} \beta_i \langle V_k(x,y), V_k(x,y) \rangle_{\Omega_i}$$

$$y_k = \frac{z_k^2(t_f) - z_k^2(0)}{2\sigma_k^2}$$

No knowledge about the thermal parameters at the interfaces is required !!

Noise & bias considerations

→ Diffusivities estimated by

$$\begin{bmatrix} \hat{\alpha}_1 \\ \vdots \\ \hat{\alpha}_p \end{bmatrix} = \mathbf{M}^{-1}\mathbf{y}$$

$$\mathbf{y} = \begin{bmatrix} \Delta\tilde{z}_1^{\ 2} / \tilde{\sigma}_1^2 + \sum_{i=1}^{2} \hat{\beta}_i \langle V_1(\mathbf{x}), V_1(\mathbf{x}) \rangle_{\Omega_i} \\ \Delta\tilde{z}_2^{\ 2} / \tilde{\sigma}_2^2 + \sum_{i=1}^{2} \hat{\beta}_i \langle V_2(\mathbf{x}), V_2(\mathbf{x}) \rangle_{\Omega_i} \\ \cdots \\ \Delta\tilde{z}_p^{\ 2} / \tilde{\sigma}_p^2 + \sum_{i=1}^{2} \hat{\beta}_i \langle V_p(\mathbf{x}), V_p(\mathbf{x}) \rangle_{\Omega_i} \end{bmatrix} \qquad \Delta\tilde{z}_i^{\ 2} = \frac{1}{2}\tilde{z}_i^{\ 2}\Big|_{t=0}^{t_f}$$

$$\mathbf{M} = \begin{bmatrix} \langle \nabla^2 V_1(\mathbf{x}), V_1(\mathbf{x}) \rangle_{\Omega_1} & \cdots & \langle \nabla^2 V_1(\mathbf{x}), V_1(\mathbf{x}) \rangle_{\Omega_p} \\ \langle \nabla^2 V_2(\mathbf{x}), V_2(\mathbf{x}) \rangle_{\Omega_1} & \cdots & \langle \nabla^2 V_2(\mathbf{x}), V_2(\mathbf{x}) \rangle_{\Omega_p} \\ \cdots & \cdots & \cdots \\ \langle \nabla^2 V_p(\mathbf{x}), V_p(\mathbf{x}) \rangle_{\Omega_1} & \cdots & \langle \nabla^2 V_p(\mathbf{x}), V_p(\mathbf{x}) \rangle_{\Omega_p} \end{bmatrix}$$

ESTIMATIONS BY
Cumulative integral-SVD method

$$u(x,y,t) = \int_0^t T(x,y,\tau)\,d\tau$$

$$u(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) z_m(t)$$

$$\Delta T(x,y,t) = \sum_{m=1}^{\infty} V_m(x,y) \eta_m(t)$$

$$\mathbf{y} = \begin{bmatrix} \langle \eta_1, z_1 \rangle_t / \tilde{\sigma}_1^2 + \sum_{i=1}^{2} \hat{\beta}_i \langle V_1(\mathbf{x}), V_1(\mathbf{x}) \rangle_{\Omega_i} \\ \langle \eta_2, z_2 \rangle_t / \tilde{\sigma}_2^2 + \sum_{i=1}^{2} \hat{\beta}_i \langle V_2(\mathbf{x}), V_2(\mathbf{x}) \rangle_{\Omega_i} \\ \cdots \\ \langle \eta_p, z_p \rangle_t / \tilde{\sigma}_p^2 + \sum_{i=1}^{2} \hat{\beta}_i \langle V_p(\mathbf{x}), V_p(\mathbf{x}) \rangle_{\Omega_i} \end{bmatrix}$$

ESTIMATIONS BY Modal-SVD method

$$\alpha_i \begin{bmatrix} \nabla^2 V_1(x,y) \\ \nabla^2 V_2(x,y) \\ \vdots \end{bmatrix} - \beta_i \begin{bmatrix} V_1(x,y) \\ V_2(x,y) \\ \vdots \end{bmatrix} = \mathbf{A} \begin{bmatrix} V_1(x,y) \\ V_2(x,y) \\ \vdots \end{bmatrix}$$

$$\mathbf{A} = \{a_{mi}\}$$

$$a_{mj} = \frac{1}{\sigma_j^2} \left\langle \dot{z}_m(t) z_j(t) \right\rangle_t$$

+ Robin conditions & adiabatic BC

$$\begin{bmatrix} \phi_1(x,y) \\ \phi_2(x,y) \\ \vdots \end{bmatrix} = \mathbf{P}^t \begin{bmatrix} V_1(x,y) \\ V_2(x,y) \\ \vdots \end{bmatrix}$$

$$\mathbf{Q}^t \mathbf{A} \mathbf{P} = \mathbf{\Sigma}$$

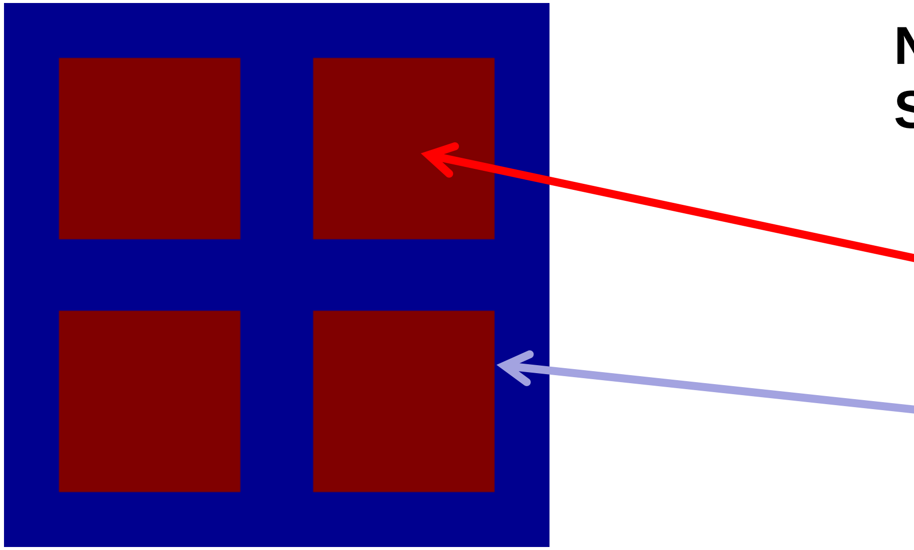
$$\mathbf{P}^t \mathbf{Q} = \mathbf{I}; \quad \mathbf{Q}^t \mathbf{P} = \mathbf{I}; \quad \mathbf{\Sigma} = diag\{\lambda_m\}$$

Set of independent PDE

$$\begin{array}{ll} \alpha_i \nabla^2 \phi_m(x,y) - \beta_i \phi_m(x,y) = \lambda_m \phi_m(x,y) & (x,y) \in \Omega_i \\ \nabla \phi_m(x,y)\mathbf{n} = 0 & (x,y) \in \partial\Omega_i \\ k_i \nabla \phi_m(x,y)\mathbf{n} = h(\phi_m(x,y)\big|_i - \phi_m(x,y)\big|_j) & (x,y) \in \partial\Omega_{ij} \end{array}$$

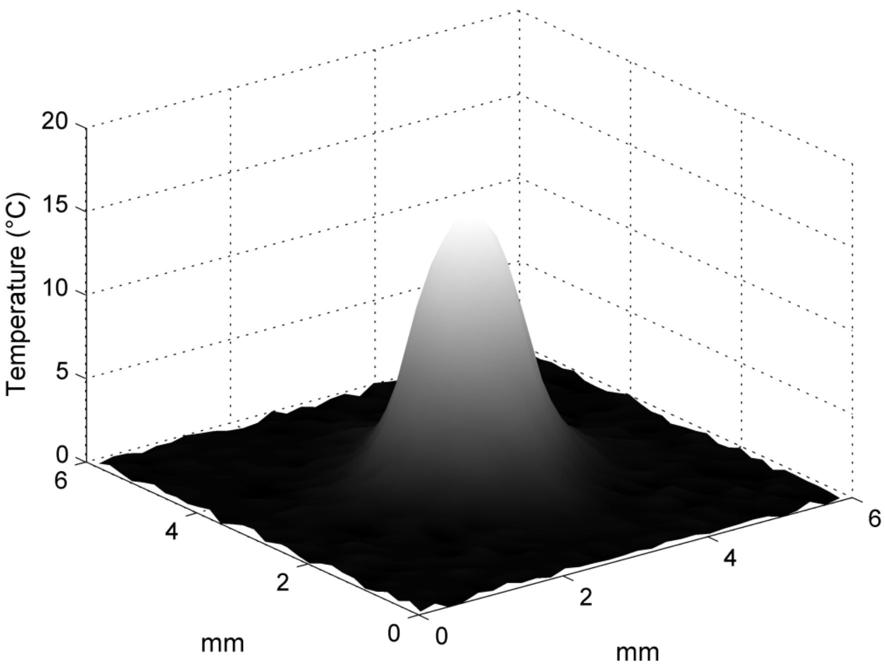
Eigenfunctions/Eigenvalues problem associated to the thermal equations

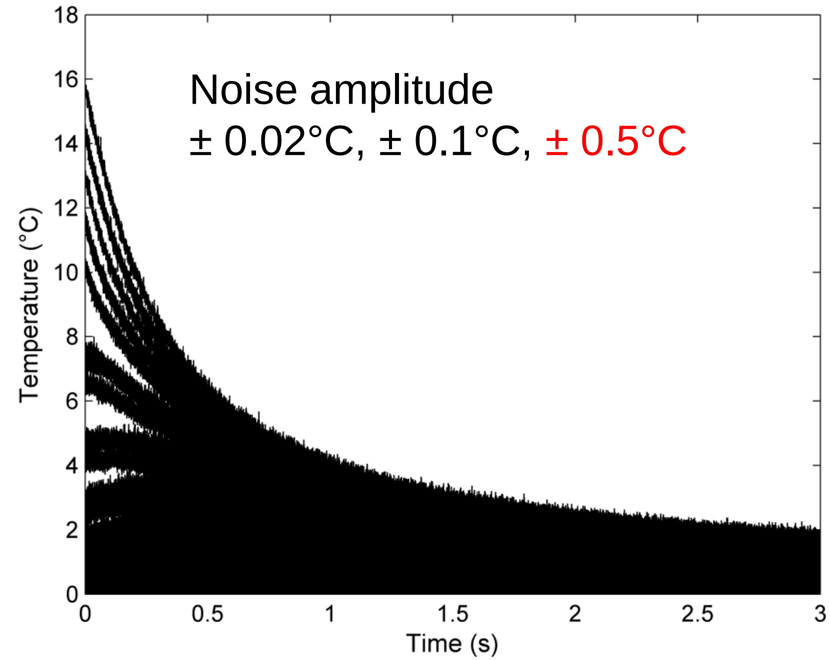
NUMERICAL TEST
SVD-based method

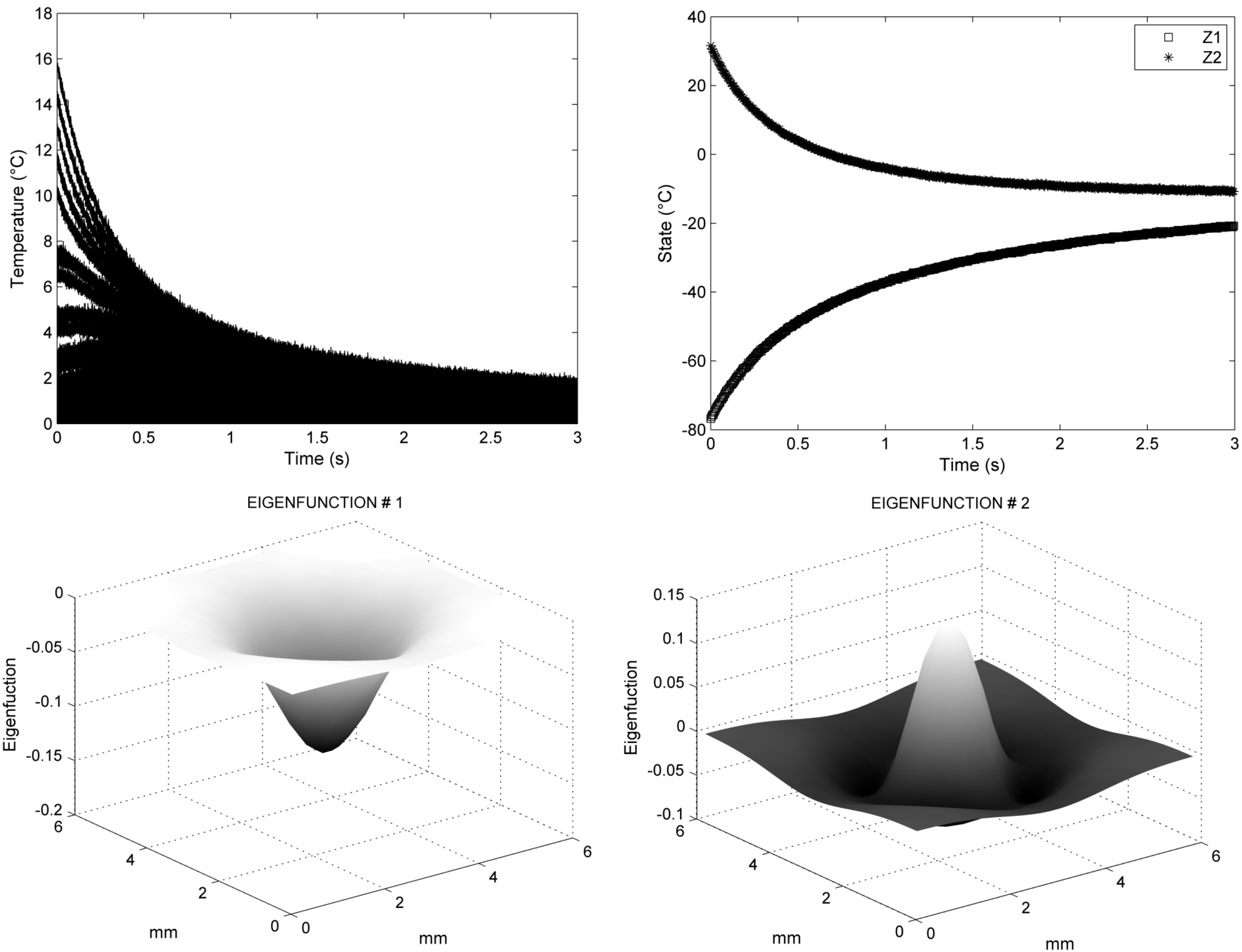


$$\alpha_1 = 1.5152 \times 10^{-7}\, m^2.s^{-1}$$
$$\beta_1 = 0.0061 s^{-1}$$

$$\alpha_2 = 5.0505 \times 10^{-7}\, m^2.s^{-1}$$
$$\beta_2 = 0.0202 s^{-1}$$







Temperature (°C)
Time (s)
State (°C)
Z1
Z2
EIGENFUNCTION # 1
EIGENFUNCTION # 2
Eigenfuction
mm

Table 3. Estimated values for thermal parameters. True values are
$\beta_1 = 0.0061\, s^{-1}$, $\alpha_1 = 1.1552 \times 10^{-7}\, m^2.s^{-1}$, $\beta_2 = 0.0202\, s^{-1}$ and $\alpha_2 = 5.0505 \times 10^{-7}\, m^2.s^{-1}$.

	Noise amplitude $(°C)$		
	± 0.50	± 0.10	± 0.02
$\hat{\beta}_1\ (s^{-1})$	0.0062	0.0061	0.0061
$\lvert(\beta_1 - \hat{\beta}_1)/\beta_1\rvert \times 100$	2.25	0.58	0.09
$\hat{\beta}_2\ (s^{-1})$	0.0208	0.0203	0.0202
$\lvert(\beta_2 - \hat{\beta}_2)/\beta_2\rvert \times 100$	2.77	0.57	0.09
$\hat{\alpha}_1\ (\times 10^{-6} m^2 s^{-1})$	0.1524	0.1526	0.1511
$\lvert(\alpha_1 - \hat{\alpha}_1)/\alpha_1\rvert \times 100$	0.60	0.73	0.23
$\hat{\alpha}_2\ (\times 10^{-6} m^2 s^{-1})$	0.5001	0.5049	0.5044
$\lvert(\alpha_2 - \hat{\alpha}_2)/\alpha_2\rvert \times 100$	1.00	0.03	0.12

Many other tests required

Microstructure
(phase fractions, topology, sizes, etc.)
Thermal properties (values & contrast)
Initial thermal field

Testing CI-SVD method

On-going research with Modal-SVD approach

Other open questions

- How to deal with low resolution images
- Temperature dependent properties

- Sensitivity to the interface location
- Retrieving interfaces at the under-pixel scale
- Estimation of thermal properties at the interfaces
- Point dependent thermal properties (ex: $\alpha(x,y)$)
- Initial thermal field providing maximum sensitivity (experiment design)

E. Palomo del Barrio & Jean-Luc Dauvergne, Chapter 14: Karhunen-Loève decomposition for data, noise, and model reduction in inverse problems, pp. 507-539, in Thermal Measurements and Inverse Techniques, Publisher: CRC Press, Editors: H. RB. Orlande, O. Fudym, D. Maillet, R.M. Cotta, 756 pages, 2011.