Inverses problems and Regularized Solutions

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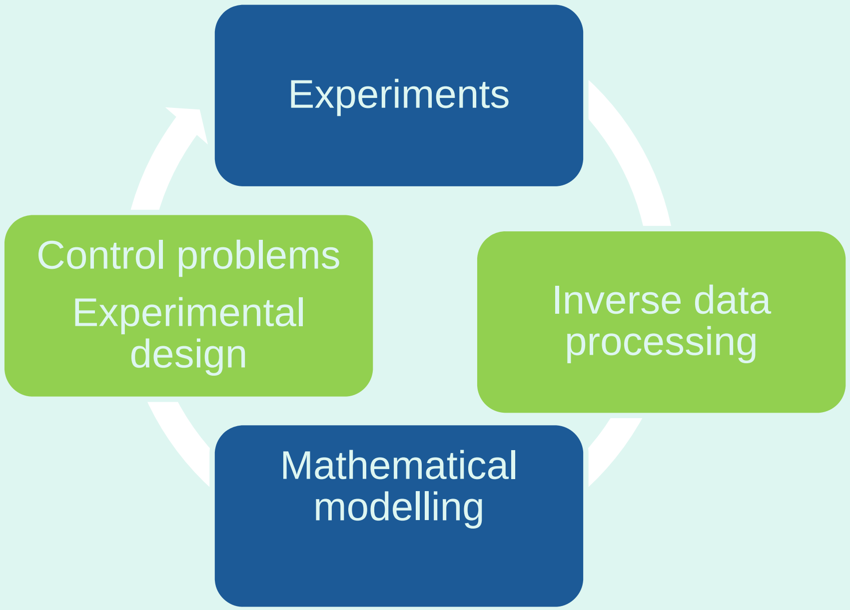
Introduction(1)

Resolution of Inverse (Heat Transfer) Problems

= a specific methodology based on the development of interactive computational-experimental process



Introduction(2)

- Inverse (heat transfer) problems are ill-posed :

- Solutions (theoretical and numerical)
- might not exist for all data
- might be not unique
- might be unstable under data perturbations

- Specific Regularization methods are required

Introduction (3)

- This lecture will be devoted to present and to illustrate the need of regularization for solving Inverse problems
- And how to do it in practice

- It will be organized in three parts

Content

- Introduction
- **Inverse Problems– 3 Basic examples of linear Ill-posed problems**
  - An initial state problem
  - A input heat source problem (semi-infinite body)
  - A 2-D stationnary heat source problem
- **Mathematical Analysis of the linear inverse problem**
  - The Singular Value Decomposition (SVD) approach
  - Quasi-solutions: existence and uniqueness conditions
  - Stability condition and regularized solutions-
- **Regularization processes and Stability condition**
  - Regularization by truncation
  - Regularization by parametrization
  - Regularization by penalization
  - Iterative Regularization
- **Conclusions**

First part: Ill-posed problems

- Example -1: initial state problem

State equations

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \quad 0 \langle x \langle 1, \quad 0 < t < t_f$$

$$T(0,t) = T(1,t) = 0, \quad 0 < t < t_f$$

$$T(x,0) = U(x), \quad 0 < x < 1$$

Output equation

$$Y(x) = T(x,t_f), \quad 0 < x < 1$$

$$\mathbf{U} = \mathbf{Y} = \mathbf{L}^2(0,1)$$

Direct problem:
Linear mapping

$$U \in \mathbf{U} \rightarrow Y \in \mathbf{Y} = AU$$

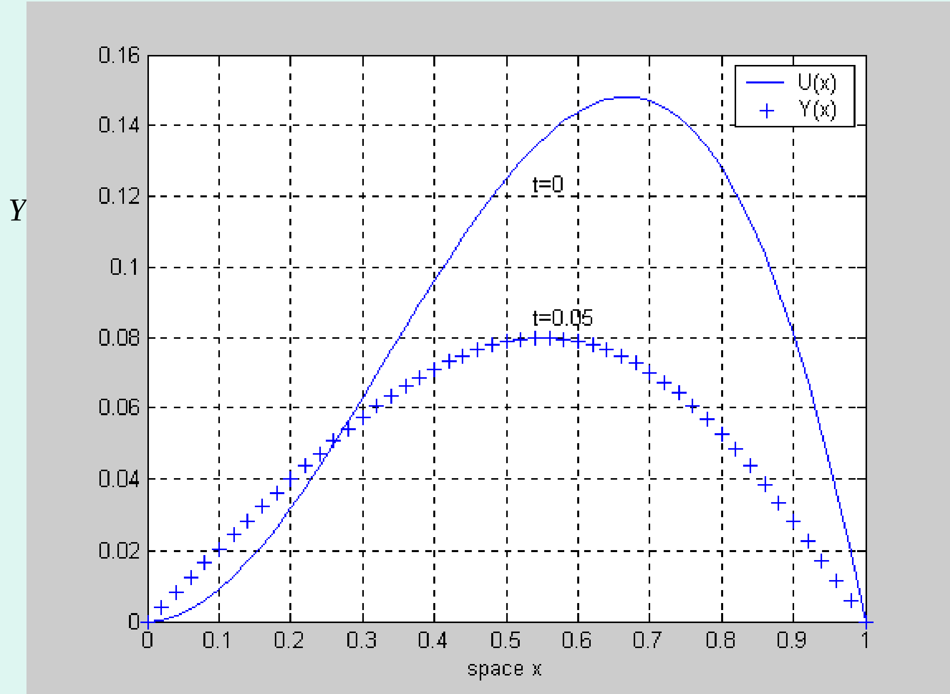
Inverse problem:

$$Y \in \mathbf{Y} \Rightarrow U = A^{-1}Y\,?$$

Example -1: initial state problem

Numerical results of the Direct Problem

$$U(x) = x^2(1 - x)$$



• Example1 : initial state problem

The solution of the direct problem is $$T(x,t) = \sum_{n=1}^{\infty} c_n . e^{n^2 \pi^2 t} \varphi_n(x)$$

with $\varphi_n(x) = \sqrt{2} \sin(n \pi x)$ = Set of orthogonal functions

so $Y(x) = \sum_{n=1}^{\infty} c_n . e^{-n^2 \pi^2 t_f} \varphi_n(x)$ and $U(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$

$$Y(x) = \sum_{n=1}^{\infty} \langle U, \varphi_n \rangle e^{-n^2 \pi^2 t_f} \varphi_n(x)$$

because $\langle \varphi_n, \varphi_m \rangle = \delta_{nm}$

The *solution of the inverse problem* is

$c_n = e^{n^2 \pi^2 t_f} \langle Y, \varphi_n \rangle$

$$U = \mathrm{A}^{-1}(Y) = \sum_{n=1}^{\infty} \langle Y, \varphi_n \rangle e^{n^2 \pi^2 t_f} \varphi_n(x)$$

• Example -1: initial state problem

The operator *A* is linear, then for any output error $\delta Y_\varepsilon = Y_\ell - Y$

The resulting variation of the solution will be $\delta U = \sum^{\infty} e^{n^2\pi^2 t_f} \langle \delta Y_\varepsilon, \varphi_n \rangle \varphi_n$

Suppose (for simplicity) $\delta Y_\varepsilon(x) = \varepsilon\sqrt{2}\sin(N\pi x)$ $\quad \|\delta Y\|_{\mathbf{Y}} = \varepsilon$

Then $$\delta U = \varepsilon\, e^{N^2\pi^2 t_f} \varphi_N, \quad \text{and} \quad \|\delta U\|_{\mathbf{U}} = e^{N^2\pi^2 t_f} \|\delta Y\|_{\mathbf{Y}}$$

any arbitrarily small output error,
may induce a great variation on the solution

The stability condition is violated,
this inverse initial state heat conduction problem is ill-posed

Example -1: initial state problem

Modeling equations

State equations

$$\frac{\partial T}{\partial t}(x,t) = \frac{\partial^2 T}{\partial x^2}(x,t), \quad x > 0, \quad 0 < t < t_f$$

$$\frac{\partial T}{\partial x}(x=0,t) = U(t), \quad 0 < t < t_f$$

$$T(x,0) = 0 \quad x > 0$$

Output equation

$$Y(t) = T(x_c, t), \quad 0 < t < t_f$$

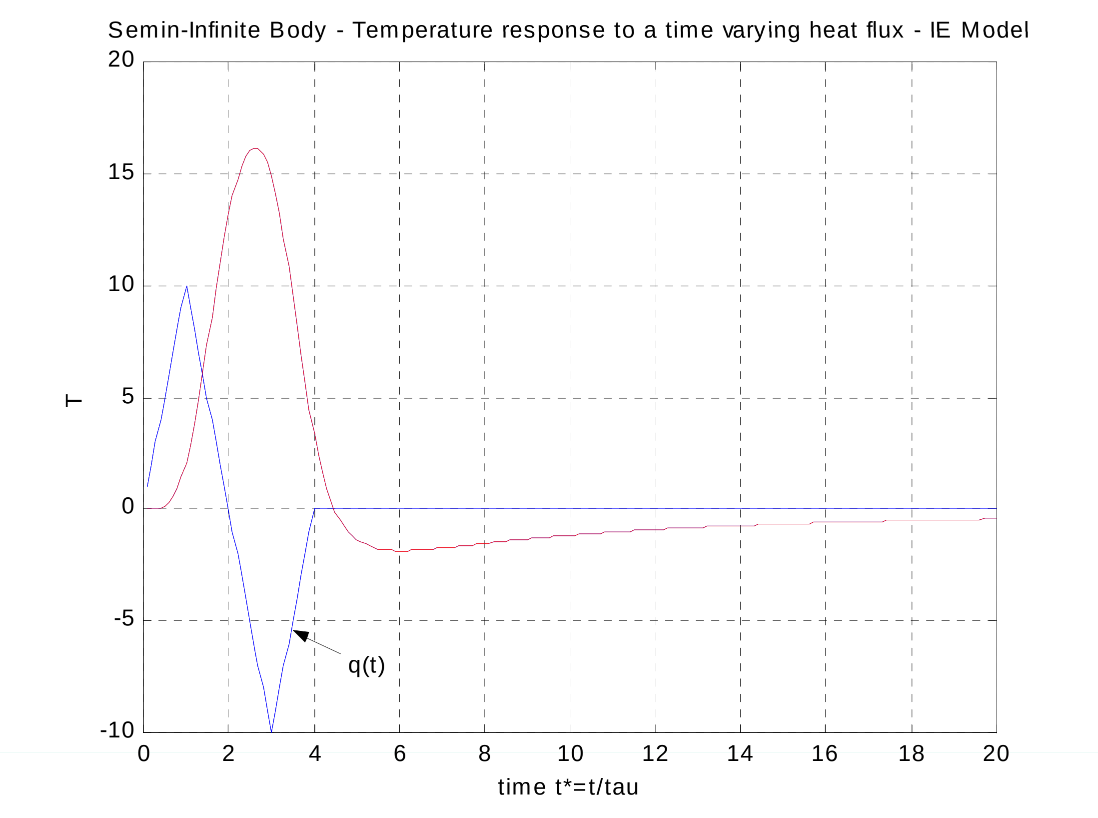
$$\mathbf{U} = \mathbf{Y} = \mathbf{L}^2(0, t_f)$$

Direct problem:
Linear mapping
(time convolution)

$$U \in \mathbf{U} \to Y \in \mathbf{Y} = AU$$

Example 2 : heat source problem (semi infinite body)

$$Y(t) = A(U)(t) = \int_{à}^{t} f_s(t-\tau)U(\tau)d\tau,\ t \in (0,t_f)$$



The impulse response

$$\frac{1}{k}\left(\frac{\alpha}{\pi t}\right)^{0.5} \exp\left(-\frac{x_s^2}{4\alpha t}\right) = f_s(t)$$

Example 3 : heat source problem
(2-D steady state)

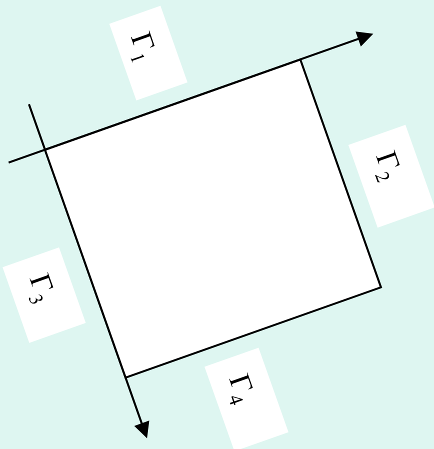
State equations

$$\Delta T = 0 \quad \text{dans } \Omega$$

$$\mathrm{T}\Big|_{\Gamma_1} = T_w$$

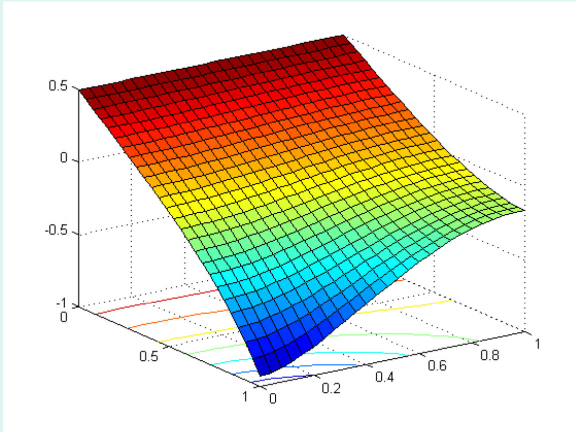
$$\left.\frac{\partial T}{\partial n}\right|_{\Gamma_2 \cup \Gamma_3} = 0$$

$$-\lambda \left.\frac{\partial T}{\partial n}\right|_{\Gamma_4} = q$$



$$T_1 = 0.5, 0 < y < 1$$

$$q(y) = q_0(\sin(\frac{\pi y}{2}) - 1), \quad 0 < y < 1$$



Output equation

$$Y = T\Big|_{\Gamma_3}$$

Example 3 : heat source problem

(2-D steady state)

After standard discretization of the spatial variable, the resulting state and output equations take the form

$$q = \left[q(y_i)\right]_{i=1}^{n} \in \mathbf{R}^n$$

$$[A]T = [B_1]*T_w + [B_4]q$$

$$T_w = [T_i]_{i=1}^{n1}$$

$$Y = [C]T$$

$$Y = \left[Y(x_j)\right]_{j=1}^{m} \in \mathbf{R}^m$$

$$Y = [C][A]^{-1}([B_1]*T_w + [B_4]q)$$

$$\delta Y = [C][A]^{-1}[B_4]\delta q = [X]\delta q$$

the numerical solution
of the inverse problem
will require the inversion of $[X]$
= the sensitivity matrix

Second Part: Linear Inverse Problem Analysis

- The three conditions of Hadamard (existence, uniqueness, stability) are investigated for the linear inverse problem in the finite dimensional case.

- The mathematical analysis will show that the concept of quasi-solution allows to satisfy the question of existence, but the uniqueness and stability conditions will remain unsatisfied.

- Then some regularization is needed to build a unique and stable quasi-solution.

Singular Value Decomposition (SVD) approach

The linear inverse problem (finite dimensional case) consists in finding

$U \in \mathbf{R}^n$ solution of the matrix equation $Y = A.U$

$Y \in \mathbf{R}^m$ and ***A*** matrix (*m x n)* are given

When the matrix is rectangular , *m ≠ n,*
the analysis of the Hadamard's conditions is required.

It is based on the singular value decomposition (SVD) of the matrix ***A***:

$$A = \mathbf{W} S \mathbf{V}^t$$

dim(**W**) = *m x m;* dim(**V**) = *n x n; dim(***S***)= m x n*

V and **W** are orthogonal

r = rank $(\boldsymbol{A}\,\boldsymbol{A}^t)$ = rank $(\boldsymbol{A}^t\boldsymbol{A})$ ≤ *inf (m,n)*

Numerical example *m = 3; n = 2*

[W, S , V]= svd(A)

$$S_{ij} = \begin{cases} \lambda_i, & \text{if } i = j = 1,..,r \\ 0, & \text{otherwise} \end{cases}$$

A	W	S	V
1 0 -1 2 0 1	0.1826 0.8944 -0.4082 -0.9129 0 -0.4082 -0.3651 0.4472 0.8165	2.4495 0 0 1.0000 0 0	0.4472 0.8944 -0.8944 0.4472

Find ***U: Y = W S V^t U***

or ***W^tY= W^t W S V^t U = S V^t U***

Let us introduce the new variables

$$Z = \mathbf{W}^t Y \in R^m \qquad X = \mathbf{V}^t U \in R^n$$

The linear inverse problem becomes: Find ***X : Z = S X***

Consequently, there are *m* algebraic equations to determine the *n* components of the vector solution *X*

$$\begin{cases} \lambda_i X_i = Z_i, i = 1,..,r \\ \sum_{j=r+1,..,n} 0 X_j = Z_i, i = r+1,..,m \end{cases}$$

The condition of existence is clearly $Z_i = 0, i = r+1,..,m$

$Z = \mathbf{W}^t Y \quad \Rightarrow Z_i = \left\langle W_i^*, Y \right\rangle_{\mathbf{R}^n}$ Im(***A***)= $\left\{ W_i^* \in \mathbf{R}^m, i = 1,..r \right\}$

Then the *existence condition* to the solution of the inverse problem is characterized by the orthogonal property equations :

$$\boxed{Y \in \mathrm{Im}(A) \Leftrightarrow \left\langle W_i^*, Y \right\rangle = 0, i = r+1,..,m}$$

The *uniqueness condition* is clearly *r = n* , which is possible only if $m \geq n$

Quasi-solutions

To satisfy the existence condition

- the least squares criterion is introduced $$J(\xi) = \left\| A\xi - Y \right\|^2_{\mathbf{R}^m}$$
- and the quasi-solutions are determined by $$U = \arg\min_{\xi \in \mathbf{R}^m} J(\xi)$$

Which gives: $A^t A U = A^t Y$

or $S^t S X = S^t Z$

Consequently, there are now *n* algebraic equations to determine the *n* components of the vector solution *X*

$$\begin{cases} \lambda_i^2 X_i = \lambda_i Z_i, i = 1,..,r \\ \sum_{j=r+1,..,n} 0 X_j = \sum_{j=r+1,..,m} 0 Z_i, i = r+1,..,n \end{cases}$$

the *existence condition* is always satisfied !

Numerical example *m = 3; n = 2; r = 2*

A	W	S	V
1 0 -1 2 0 1	0.1826 0.8944 -0.4082 -0.9129 0 -0.4082 -0.3651 0.4472 0.8165	2.4495 0 0 1.0000 0 0	0.4472 0.8944 -0.8944 0.4472

$$U = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow Y = AU = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow Z = W'*Y = \begin{bmatrix} -2.1909 \\ 0.4472 \\ 0 \end{bmatrix} = SX$$

$$\Rightarrow X = \begin{bmatrix} -0.8944 \\ 0.4472 \end{bmatrix} \Rightarrow U = V*X = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

With MATLAB:

U=A\Y

U=inv(S'*S)*S'*Y

U=pinv(A)*Y

But if $r < n$, there are an infinite set of quasi-solutions !

$$X = \sum_{i=1,..r} \frac{Z_i}{\lambda_i} \mathbf{V}_i + \sum_{i=r+1,..,n} c_i \mathbf{V}_i$$

$\{c_i, i = r+1,..,n\}$ Is a set of arbitrary constant values

One way to satisfy the *uniqueness condition,* consists

- in introducing some *a priori* estimate X_{est}
- in determining $\{c_i, i = r+1,..,n\}$ in order to get the closest solution X^*, i.e.
- in minimizing the distance $\left\| X^* - X_{est} \right\|$

So they satisfy the orthogonal properties : $\left\langle \mathbf{V}_i, X_i^* - X_{est,i} \right\rangle = 0, i = r+1,..,n$
And <u>the unique quasi-solution</u> is therefore

$$\boxed{X^* = \sum_{i=1,..r} \frac{Z_i}{\lambda_i} \mathbf{V}_i + \sum_{i=r+1,..,n} X_{est,i} \mathbf{V}_i}$$

Stability condition and regularized solutions- Matrix ***A*** Squared

- The matrix ***A*** is squared and non singular, *m = n*
- The data are corrupted by an additive noise : $Y = Y_{ex} + \delta Y$ $\quad \left\| \delta Y \right\|_{\mathbf{R}^m} \leq \varepsilon$
- The matrix ***W^t*** is orthogonal, then $\left\| \delta Y \right\|_{\mathbf{R}^m} = \left\| \delta Z \right\|_{\mathbf{R}^m} \leq \varepsilon$
- The error corrupts only the component *k*, $\delta Z = \mathbf{W}^t \delta Y = \varepsilon W_k$

The quasi-solution is
$$X = \sum_{i=1,..n} \frac{Z_i}{\lambda_i} \mathbf{V}_i$$

And the error generated is
$$\delta X = \frac{\varepsilon}{\lambda_k} V_k$$

the relative error variation is then
$$\boxed{\frac{\left\| \delta U \right\|_{\mathbf{R}^n}}{\left\| \delta Y \right\|_{\mathbf{R}^m}} = \frac{\left\| \delta X \right\|_{\mathbf{R}^n}}{\left\| \delta Z \right\|_{\mathbf{R}^m}} = \frac{1}{\lambda_k}}$$

Stability condition and regularized solutions- Matrix ***A*** Squared

This result means that an error ε

on the component Z_n of the data

creates a perturbation on the solution which is λ_1/λ_n
times greater than the same error
on the component Z_1

This ratio is called *cond(A)* the *condition number* of the matrix *A*.

In practice, a large value for this ratio means
• that the solution will be very sensitive to the possible data errors
• and that some regularization process has to be performed

Stability condition and regularized solutions- Matrix *A* Rectangular

The sensitivity of the quasi-solutions to output data errors is characterized by the condition number of the matrix equation

$$\left[S^t S\right]\delta X = S^t \delta Z$$

When $m \geq n$ and $r = n$

$$cond(S^t S) = \lambda_1^2 / \lambda_{r=n}^2$$

Third Part: Regularization processes and Stability condition

General ideas

There are several ways to regularize the inversion process,
i.e. to make the quasi-solution less sensitive to the data errors
and to satisfy the stability condition.

All of them consists in adding some *a priori* information on the solution to be determined

Two great approaches are briefly presented and illustrated:

- One approach consists in searching for a quasi-solution which satisfies some *a priori constraints*
- An other approach is based on the *"penalization"* of the L-S criterion

Different possibilities are available for defining these constraints, the most usuals are

- the *truncation* of the basis $\left\{ V_i^* \in \mathbf{R}^n, i = 1,..r \right\}$ given by the SVD of the matrix ***A***

$$X = \sum_{i=1,..n} \frac{Z_i}{\lambda_i} \mathbf{V}_i \qquad Z_i = \left\langle W_i^*, Y \right\rangle_{\mathbf{R}^n}$$

- the *parametrization* of the solution $U = \sum_{i=1,p} U_i \omega_i$

Where the set of basis vectors $\left\{ \omega_i \in \mathbf{R}^p, i = 1,..p \right\}$ Is a priori given

Regularization by truncation

The idea is to constrain the quasi-solution

to belong to the sub-space $\chi = \left\{ V_i \in \mathbf{R}^n, i = 1,..p \right\}$

The "regularizing parameter" is then the order of the truncation $p < n$

for which the condition number will become "acceptable".

In practice, this truncation means that the components of the data *Y*

corresponding to the vectors $\left\{ W_i \in \mathbf{R}^m, i = p+1,..r \right\}$

Are "eliminated" !

and the regularized solution is: $\boxed{X_c = \sum_{i=1,..p} \frac{\langle Z, W_i \rangle}{\lambda_i} \mathbf{V}_i}$

Consequently, the inversion of the modified data instead of the original one

will introduce a *bias*, i.e. a systematic error on the computed solution.

Regularization by truncation_ Numerical example

$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$$

$$cond(A) = \frac{30.2887}{0.0102} = 2.9841e3$$

$$Y = \left| 32 \quad 23 \quad 33 \quad 31 \right|^t$$

+

$$\delta Y = \left| 0.1 \quad -0.1 \quad 0.1 \quad -0.1 \right|^t$$

$$U = A^{-1} Y = \left| 1 \quad 1 \quad 1 \quad 1 \right|^t$$

$$\delta U = A^{-1} \delta Y = \left| 8.2 \quad -13.6 \quad 3.5 \quad -2.1 \right|^t$$

$$G = \frac{\|\delta U\|}{\|U\|} \left[\frac{\|\delta Y\|}{\|Y\|} \right]^{-1} = 2460.6$$

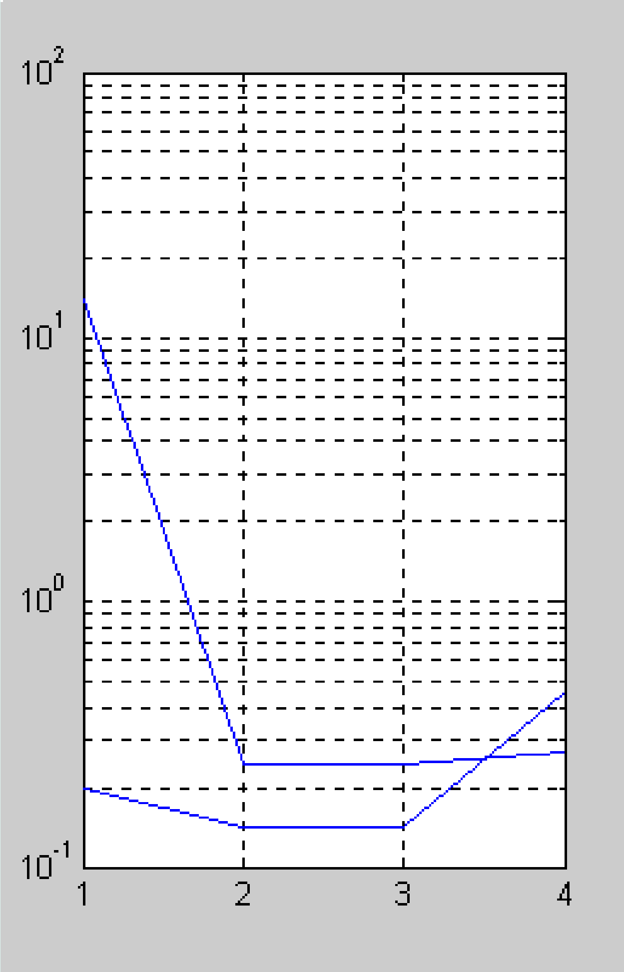
The regularized solutions build by truncation

32.0000	32.1000	32.0302	32.0201	31.7910
23.0000	22.9000	23.0156	23.0187	22.8711
33.0000	33.1000	33.0710	33.0964	33.1976
31.0000	31.1000	31.1172	31.0982	31.3313

n - p	0	1	2	3
$\left\| Y_p - Y_{exact} \right\|$	0.2000	0.1412	0.1403	0.4573

1.0000	8.0000	1.1209	1.1090	1.0496
1.0000	-10.600	0.7897	0.7934	0.7551
1.0000	3.9000	1.0397	1.0698	1.0960
1.0000	-0.7000	0.9965	0.9740	1.0344

n - p	0	1	2	3
$\left\| U_p - U_{exact} \right\|$	13.9592	0.2459	0.2452	0.2699



Regularized solutions computed for *p=1,2* are quite acceptable

(In practice, this approach is not available, the exact solution is unknown)

Regularization by parametrization

The quasi-solution is constrained to belong to a subspace, which is a priori given

Application to the 2-D inverse heat source problem

$$Y = [C][A]^{-1}([B_1] * T_w + [B_4]q) \qquad q = [q(y_i)]_{i=1}^{n} \in \mathbf{R}^n$$

$$q(y) = \sum_{j=1}^{p} U_j \omega_j(y)$$

$\{\omega_i, i = 1,..p\}$ is a set of *p* piecewise linear functions

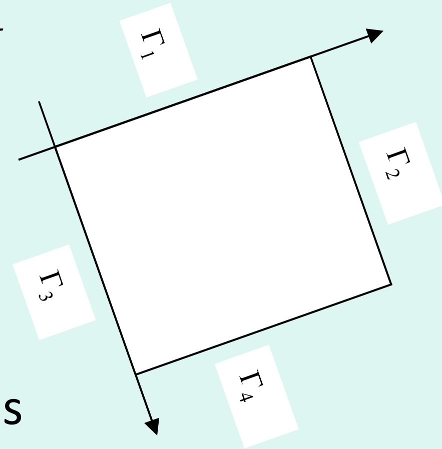
(« hat functions ») on the boundary $\Gamma_4 = [0,1]$

such that $\omega_j(y_k) = \delta_{jk}$

then $M_{ij} = \omega_j(y_i), i = 1,.., n;\ j = 1,.., p \Rightarrow q = MU$

Now the ill-conditioness of the linear inverse problem is characterized by the condition number of the new sensitivity equation :

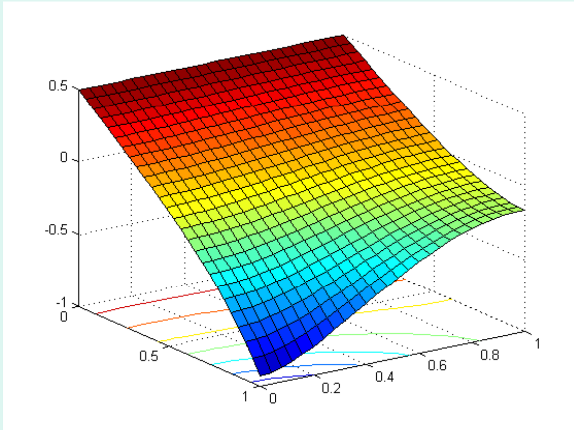
$$\delta Y = [C][A]^{-1}[B_4][M]\delta U = [X][M]\delta U$$



The previous condition number $cond\left(X^t X\right)$

Becomes $cond\left(M^t X^t XM\right)$

The temperature field, solution of the direct problem, is computed on a *m x m* spatial grid,



size*(Y) = m* , size ([X])= *m x n,*

Size*(q)* =*n, and n = m – 1*

(m nodes and *n = m-1* intervals)

m	5	8	11	16	21	26
cond(X'*X)	9.2987e+004	5.0208e+009	2.3942e+014	1.1284e+019	4.7040e+020	8.1567e+020

Condition number before parametrization – Influence of the grid size *m x m*

m	16	21	26
p=5 yp=[0; 0.25; 0.5; 0.75; 1.0]	7.5950e+008	6.5797e+008	5.8852e+008
p=4 yp=[0;0.3;0.6;1.0]	1.6898e+006	1.6087e+006	1.5596e+006

Influence of *p* on the new condition number $cond\left(M^t X^t XM\right)$ after parametrization

$cond\left(M^t X^t XM\right)$ becomes independent of the mesh size

Some desired accuracy of the numerical solution of the direct problem can be reached without increasing the unstability of the inverse problem solution

Regularization by penalization

The idea of the regularization process by *"penalization"* consists in considering a new L-S criterion which includes the *a priori* estimate X_{est} and a positive parameter so-called regularization parameter $\mu \in [0,1]$

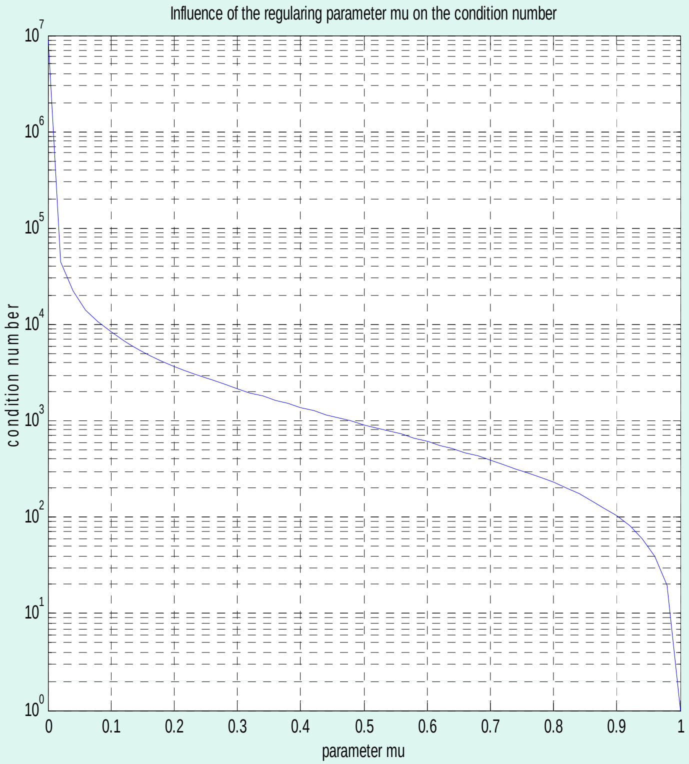
$$J_\mu(\xi) = (1-\mu)\left\| S\xi - Z \right\|^2_{\mathbf{R}^m} + \mu \left\| \xi - X_{est} \right\|^2_{\mathbf{R}^n}$$

the "regularized" quasi-solution , and its components are

$$X^*_\mu = \left[(1-\mu)S^t S + \mu I \right]^{-1} \left[(1-\mu)S^t Z + \mu X_{est} \right]$$

$$X_i^* = \frac{(1-\mu)\lambda_i Z_i + \mu X_{i,est}}{(1-\mu)\lambda_i^2 + \mu}, i = 1,..,n \qquad \boxed{\Rightarrow \delta X_i^* = \frac{(1-\mu)\lambda_i}{(1-\mu)\lambda_i^2 + \mu} \delta Z_i, i = 1,..,n}$$

Regularization by penalization_ Numerical example



$$A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}$$

$$cond(A) = \frac{30.2887}{0.0102} = 2.9841e3$$

the new condition number

$$cond\left[(1-\mu)S^t S + \mu I\right]$$

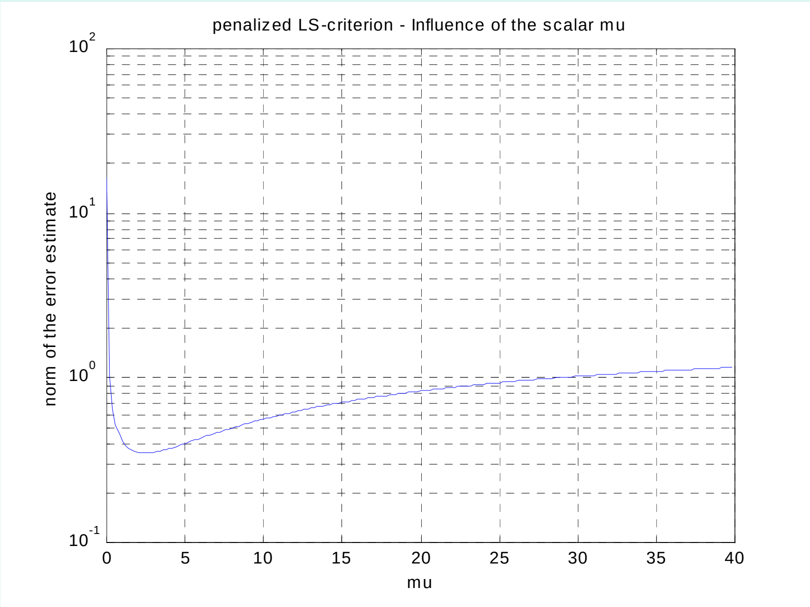
Is a decreasing function of the

regularizing parameter

Optimal value of the regularizing parameter – Numerical example

$$J_{\rho}(U) = \left\| Y_{\varepsilon} - AU \right\|^{2} + \rho \left\| U \right\|^{2} \qquad U_{est} = 0$$

$$Y = \begin{vmatrix} 32 & 23 & 33 & 31 \end{vmatrix}^{t} \quad + \quad \delta Y = \begin{vmatrix} 0.1 & -0.1 & 0.1 & -0.1 \end{vmatrix}^{t} \qquad \left\| \delta Y \right\| = \varepsilon = 0.2$$



Plot of the norm of the

error estimate $\left\| dU \right\|$

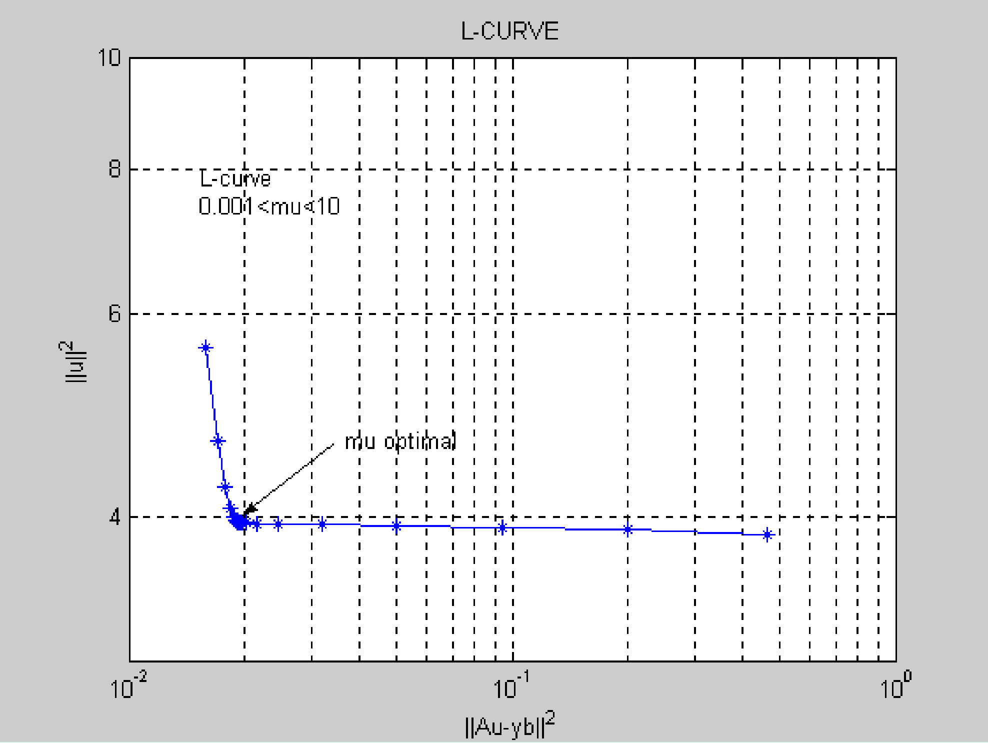
Versus the regularization

parameter $\rho = \dfrac{\mu}{1-\mu}$

The best value

$$\hat{\rho} \in [1,3] \Rightarrow \hat{\mu} \in [0.50, 0.75]$$

Optimal value of the regularizing parameter – Numerical example
The L curve method (Hansen, 1992)



ρ	$\|AU_{\varepsilon,\rho} - Y_\varepsilon\|^2$	$\|U_{\varepsilon,\rho}\|^2$
The under-regularized region		
00010	0.0159	5.5950
0.0016	0.0171	4.6548
0.0025	0.0179	4.2460
0.0040	0.0184	4.0737
0.0063	0.0188	4.0025
0.0100	0.0190	3.9735
0.0158	0.0191	3.9617
0.0251	0.0192	3.9569
0.0398	0.0193	3.9548
0.0631	0.0193	3.9537
the corner region		
0.1000	0.0194	3.9529
0.1585	0.0195	3.9521
0.2512	0.0197	3.9509
0.3981	0.0203	3.9493
0.6310	0.0216	3.9468
1.0000	0.0246	3.9431
the over-regularized region		
1.5849	0.0320	3.9374
2.5119	0.0499	3.9286
3.9811	0.0939	3.9150
6.3096	0.2014	3.8941
10.0000	0.4634	3.8619

The L-curve corner where the curvature of the log-log plot is maximised is readily visible.
It does not correspond to one specific value of the regularizing parameter but to some interval

Optimal value of the regularizing parameter – Numerical example
The discrepancy principle (Morozov- 1984; Alifanov, 1994)

$$S_\rho(X) = \|Z - SX\|^2 + \rho\|X\|^2 \Rightarrow X_\rho^* = \arg\min S_\rho(X)$$

The components of the regularized solutions are $X_{i,\rho}* = \dfrac{\lambda_i Z_i}{\lambda_i^2 + \rho}$

Let us choose the parameter such that
the LS-residual is equal to the noise level

$$\|Z - SX_\rho^*\|^2 = \varepsilon^2$$

Then it is solution of the algebraic equation

$$\phi(\rho) = \sum_i \left[\frac{\rho Z_i}{\lambda_i^2 + \rho}\right]^2 - \varepsilon^2 = 0$$

With MATLAB, we get:
Zero found in the interval: [0.54745, 1.4525].
roopt = 1.4065
U_{opt} = [1.1096 0.8013 1.0849 0.9391]

Iterative Regularization

The discrepancy principle and the conjugate gradient algorithm

For large scale linear inverse problems, or non linear problems,
direct computation of the SVD can become non efficient or impracticable

The method is iterative - at each iteration: $J(U^n) = \left\| Y_\varepsilon - AU^n \right\|^2$

1. Compute the gradient $\nabla J(U^n) = -2A^t \left[Y_\varepsilon - AU^n \right], n = 0,1,..$

2. Determine a new approximation of the solution

$$\gamma^* = \underset{\gamma > 0}{\operatorname{argmin}} J(U^n + \gamma d^n)$$

$$U^{n+1} = U^n + \gamma^n d^n$$

$$d^n = -\nabla J(U^n) + \beta^n d^{n-1}$$

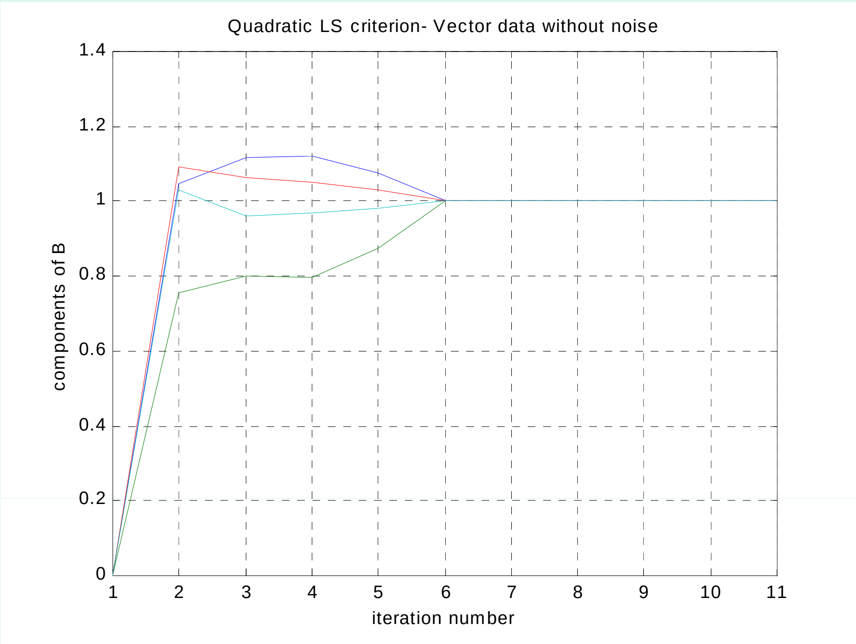
$$\beta^0 = 0; \quad \beta^n = \left\| \nabla J(U^n) \right\|^2 \Big/ \left\| \nabla J(U^{n-1}) \right\|^2$$

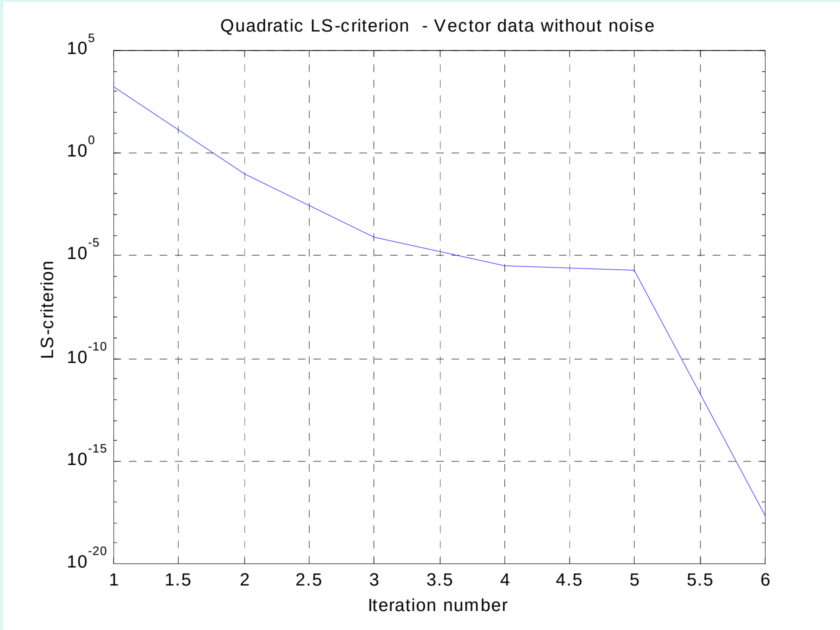
3. Use the discrepancy principle as a stopping rule $J(U^{nf}) = \left\| Y_\varepsilon - AU^{nf} \right\|^2 \approx \varepsilon^2$

The last iteration index *nf* is the regularizing parameter

Conjugate gradient algorithm– Numerical example
Exact data

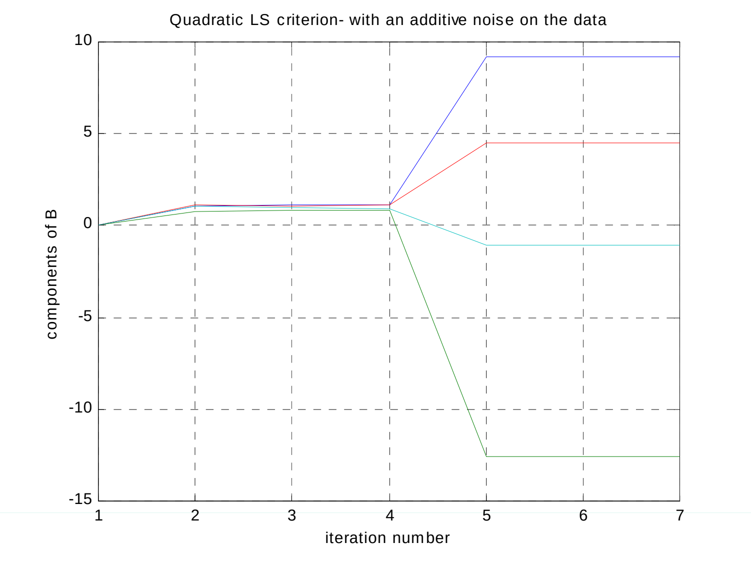
Iteration i	0	1	2	3	4	5	6
Iterative solution (Conjugate gradient algorithm)	0	1.0486	1.1179	1.1224	1.0755	1.0000	1.0000
	0	0.7543	0.7987	0.7974	0.8749	1.0000	1.0000
	0	1.0933	1.0622	1.0509	1.0314	1.0000	1.0000
	0	1.0312	0.9613	0.9698	0.9814	1.0000	1.0000

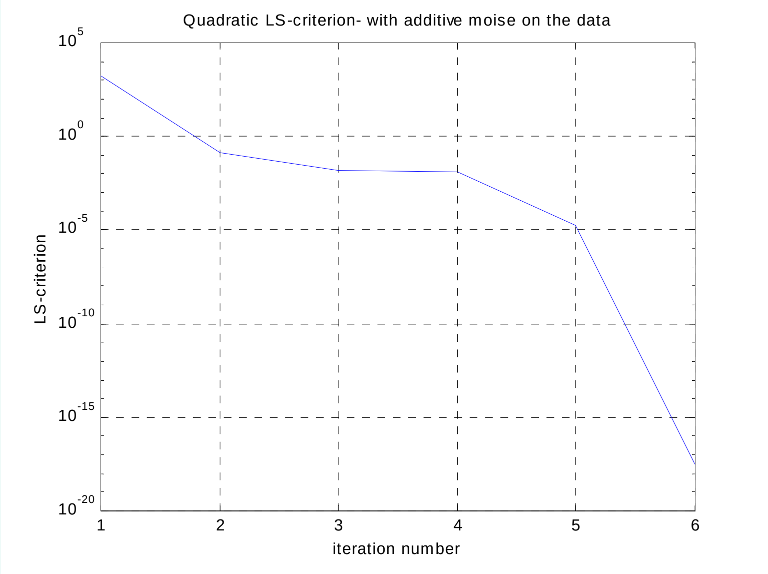




Conjugate gradient algorithm– Numerical example
Iterative regularization - Noisy data

$$\varepsilon^2 = 4.10^{-2}$$



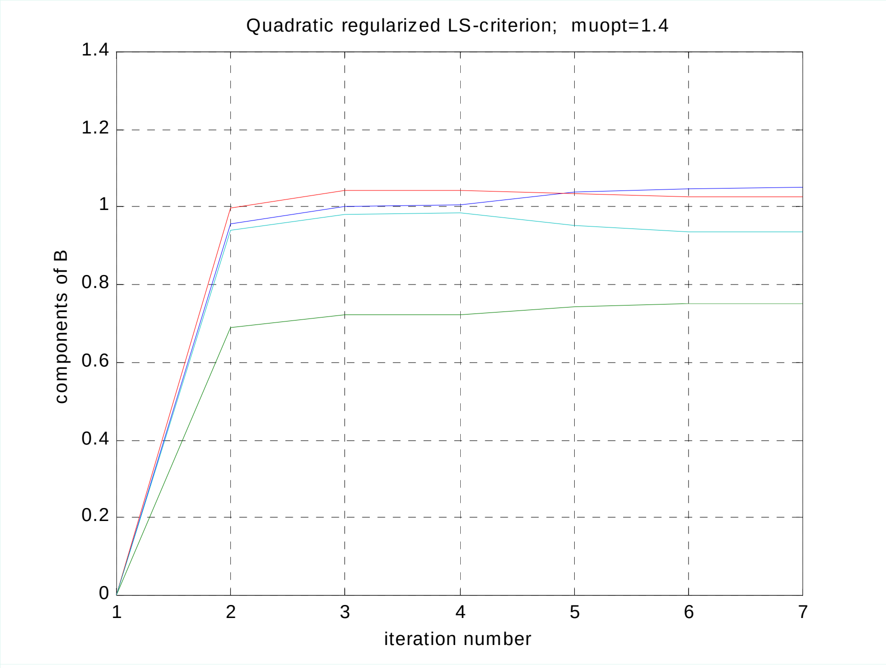


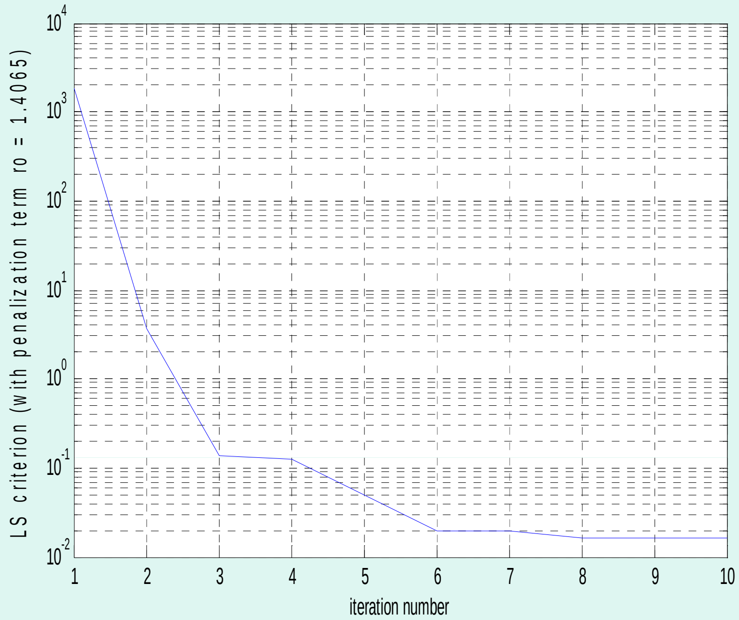
Iteration number	LS - criterion
0	1.802620000000000e+003
1	1.365119982194983e-001
2	1.634291561475751e-002
3	1.343557552068297e-002
4	1.690700995036237e-005
5	3.210230589037177e-018

Regularized solution at *nf* = 3

$$U^{(3)} = \begin{vmatrix} 1.0997 \\ 0.8117 \\ 1.1319 \\ 0.8977 \end{vmatrix}$$

Conjugate gradient algorithm– Numerical example
Regularization by penalization - Noisy data





0	0.9584	1.0018	1.0054	1.0382	*1.0485*	*1.0496*
0	*0.6894*	*0.7206*	*0.7231*	*0.7449*	*0.7515*	*0.7523*
0	0.9991	1.0429	1.0448	1.0348	1.0265	1.0268
0	0.9422	0.9827	0.9836	0.9531	0.9361	0.9358

$$S_{\hat{\rho}}(U) = \|Y_\varepsilon - SU\|^2 + \hat{\rho}\|U\|^2$$

$$\hat{\rho} = 1.4065$$

Conjugate gradient algorithm– The heat source problem
Iterative regularization - Noisy data

The model equation (time convolution)

$$Y(t) = A(U)(t) = \int_0^t f(t-\tau)U(\tau)d\tau, \quad t \in (0, t_f)$$

$$f(t) = K\sqrt{\frac{\tau}{t}}\exp(-\frac{\tau}{t}), \quad \tau = \frac{x_s^2}{\alpha}$$

The LS-criterion

$$J(U) = \frac{1}{2}\int_0^{t_f}\left[Y(t;U) - Y_\varepsilon(t)\right]^2 dt = \frac{1}{2}\left\|Y(U) - Y_\varepsilon\right\|_{\mathbf{L}^2}^2$$

The gradient computed with an adjoint variable

$$\nabla J(t;U) = \int_t^{t_f} \psi(x;U) f(x-t)dx, \ 0 < t < t_f$$

$$\psi(x) = \left[Y(x;U) - Y_\varepsilon(x)\right]$$

Conjugate gradient algorithm– The heat source problem
Iterative regularization - Noisy data

Numerical results

$$t_k = k\Delta t, \quad \Delta t = \frac{t_f}{nf} \quad k = 1,..,nf$$

$$Y_k = Y(t_k) = \Delta t \sum_{i=1}^{k} f_{k-i} U_i \ ,$$

$$\nabla J_k = \nabla J(t_k) = \Delta t \sum_{i=k+1}^{nf} f_{i-k} \psi_i$$

$$f_n = f(t_n) = \frac{1}{\sqrt{t_n}} \exp\left(-\frac{1}{\sqrt{t_n}}\right)$$

The solution $U = \left[U_k\right]_{k=1}^{nf}$

is computed over the time interval

$\left[0, t_f = 5\right] \quad \Delta t = 0.1, \quad nf = 51$

An normally distributed noise is added

to get the output noisy data Y_ε

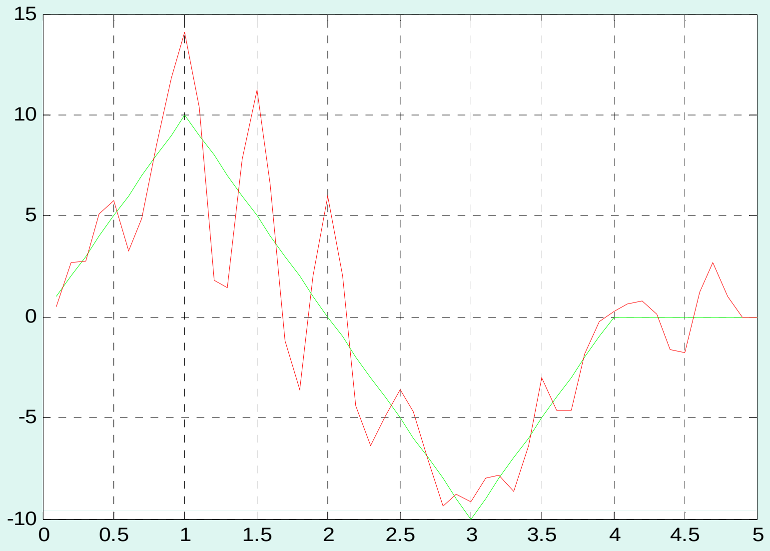
Two numerical experiments are performed with

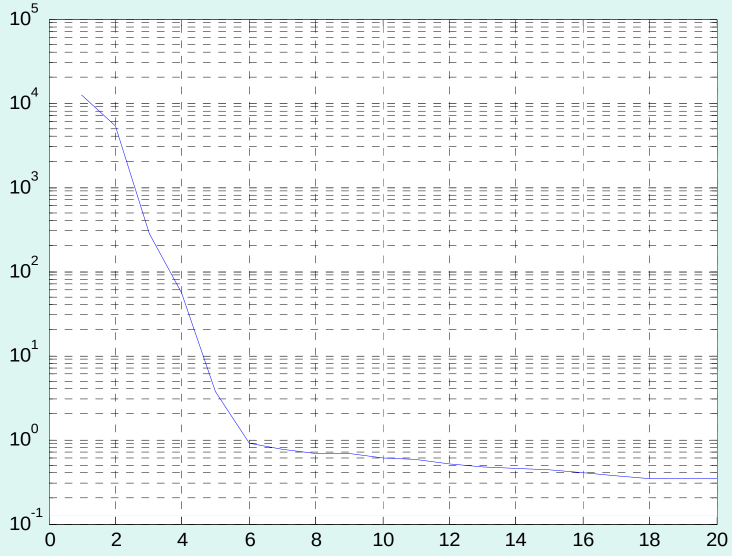
different noise level.

The initial guess : $U_k^{(0)} = 0, k = 1..nf$

Conjugate gradient algorithm– The heat source problem
Iterative regularization - Noisy data

Numerical results: too much iterations give unstables solution !



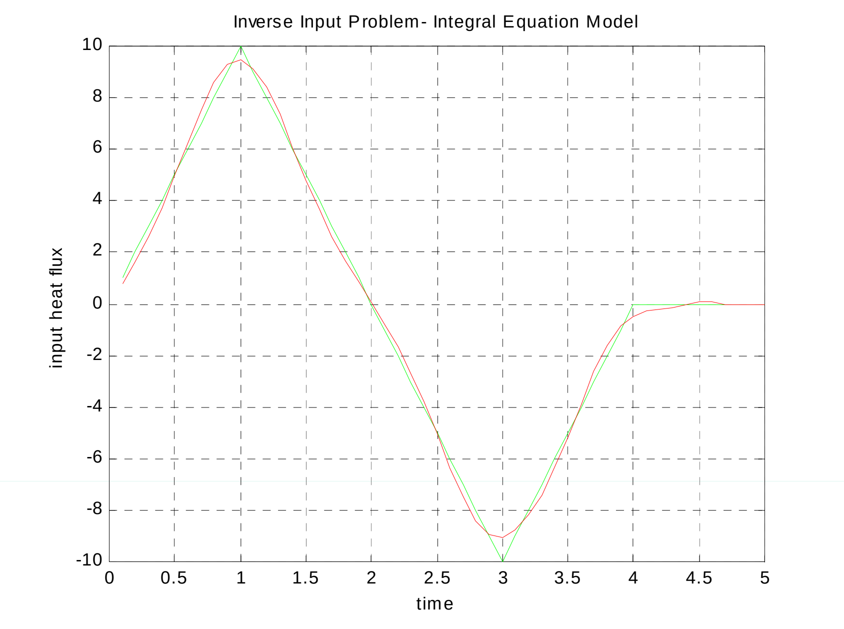


Computed (unstable) Solution at iteration nf = 20

$$\varepsilon^2 = \|\delta Y\|^2 = 2.12$$

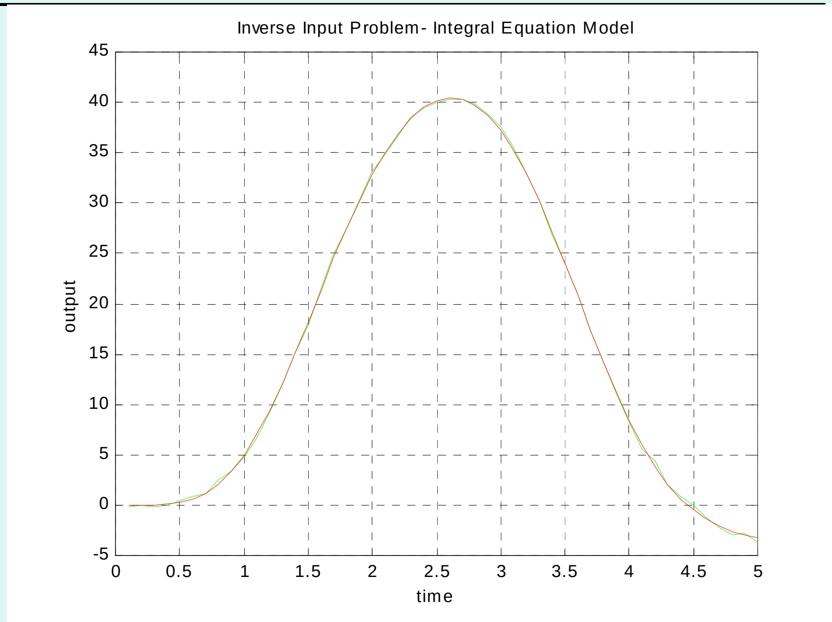
Conjugate gradient algorithm– The heat source problem
Iterative regularization - Noisy data - Numerical results

$$\varepsilon^2 = \left\|\delta Y\right\|^2 = 2.12$$



Exact input and computed solution

after 5 iterations

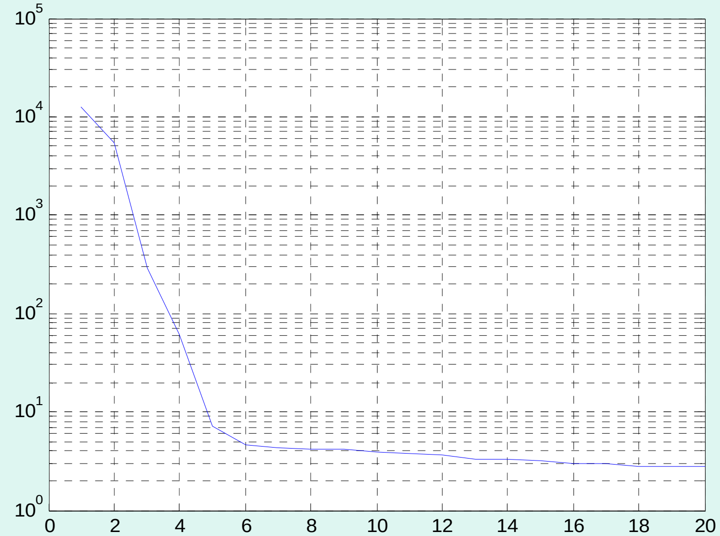


Output data and computed output

after 5 iterations

Conjugate gradient algorithm– The heat source problem
Iterative regularization - Noisy data

Numerical results: too much iterations give unstables solution !

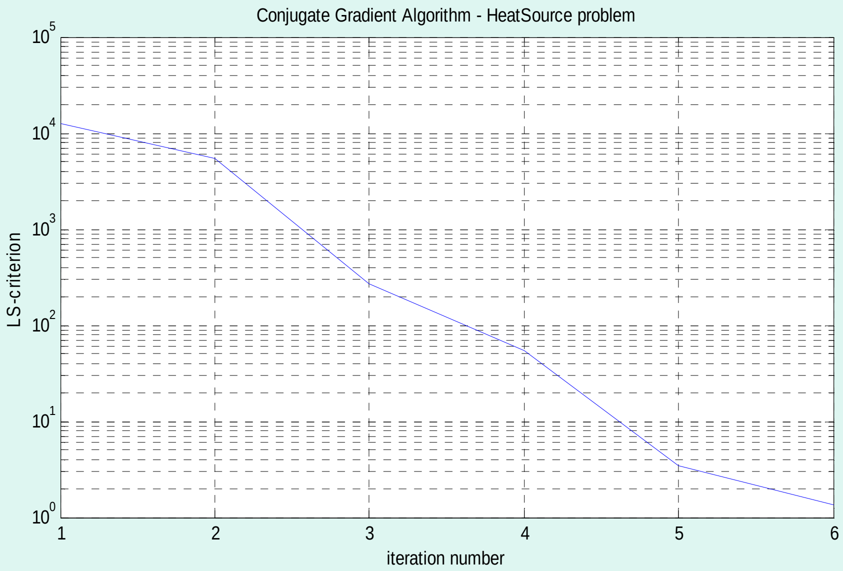


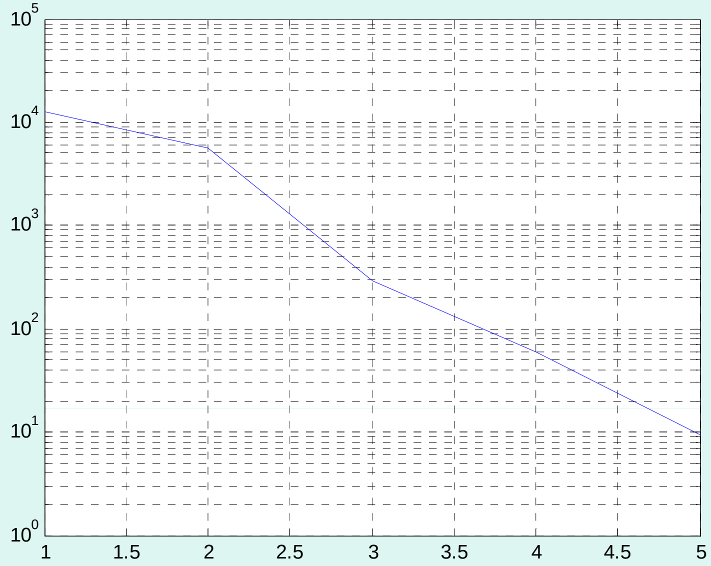
Computed (unstable) Solution at iteration nf = 20

$$\varepsilon^2 = \|\delta Y\|^2 = 10.39$$

Conjugate gradient algorithm– The heat source problem
Iterative regularization - Noisy data

Numerical results: The LS-criterion versus the iteration number





$$\varepsilon^2 = \|\delta Y\|^2 = 2.12$$

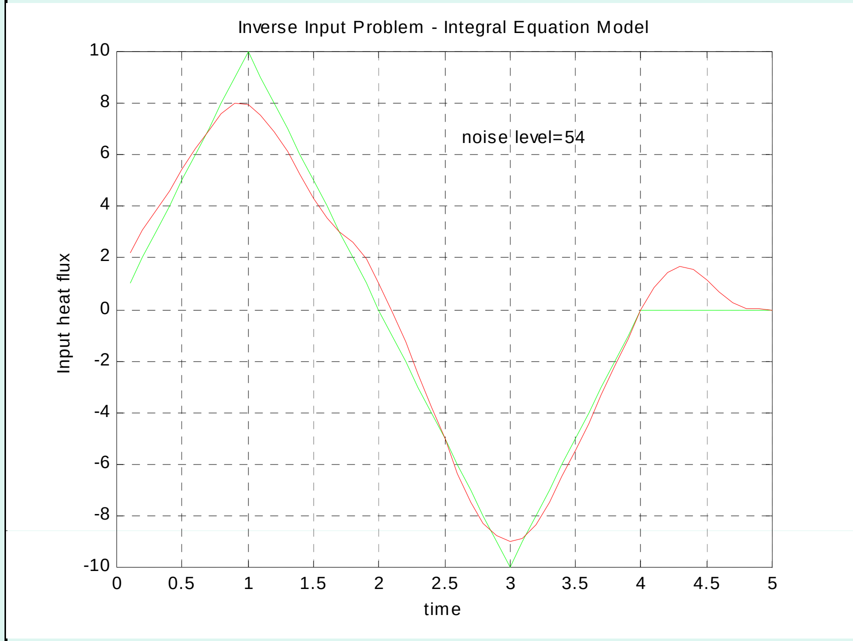
$$\varepsilon^2 = \|\delta Y\|^2 = 10.39$$

Conjugate gradient algorithm– The heat source problem

Iterative regularization - Noisy data $\varepsilon^2 = \|\delta Y\|^2 = 10.39$

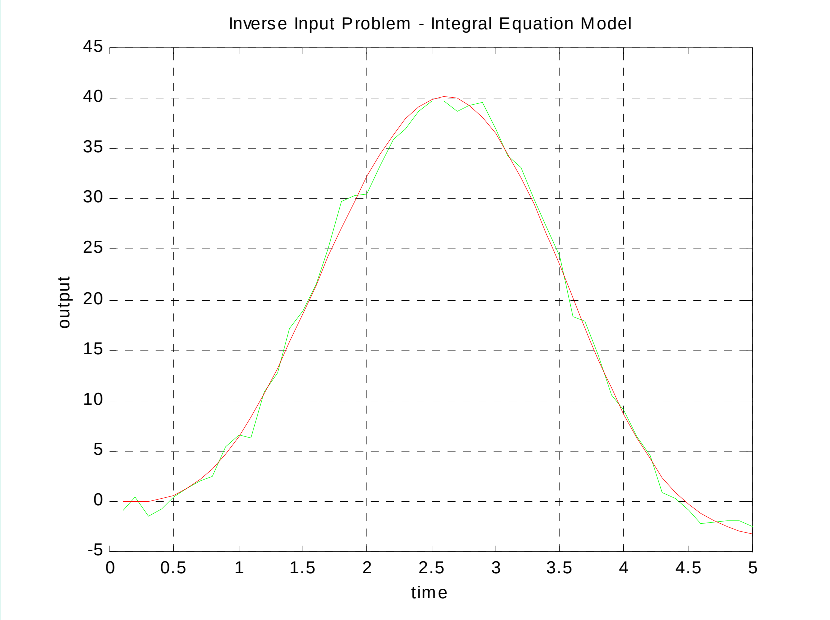
Numerical results

.



Exact input and computed solution

after 4 iterations



Output data and computed output

after 4 iterations

Conjugate gradient algorithm– The heat source problem
Iterative regularization - Noisy data

The numerical resolution of this linear inverse problem

- shows how the computation of the gradient of the LS criterion allows to construct regularized solution of an inverse input problem (formulated here with an integral equation)
- shows how the discrepancy principle is an efficient way to avoid the amplification of the data errors, and it is easy to implement with the CG algorithm
- confirms that the last iteration index is the regularizing parameter, and how its value depends on the noise level of the data to be inverted

Conclusions

1. A key feature of inverse problems is their ill-posedness. Construction of quasi-solutions by minimizing an output least square criterion is a general approach for solving.
2. Basic algebra results led to efficient algorithms for the computation of regularized solutions. They are based on the SVD analysis of the linear operator.
3. To make the quasi-solutions less sensitive to the data errors and to satisfy the stability condition, some adding *a priori* information, on the solution to be determined, has to be considered.
4. Two basic approaches were briefly presented and illustrated: a)quasi-solutions which satisfy some *a priori constraints, or b)* based on the *"penalization"* of the L-S criterion.
5. The *conjugate gradient algorithm* to be known among the most effective method to compute regularized solution according to the *discrepancy principle,* was illustrated

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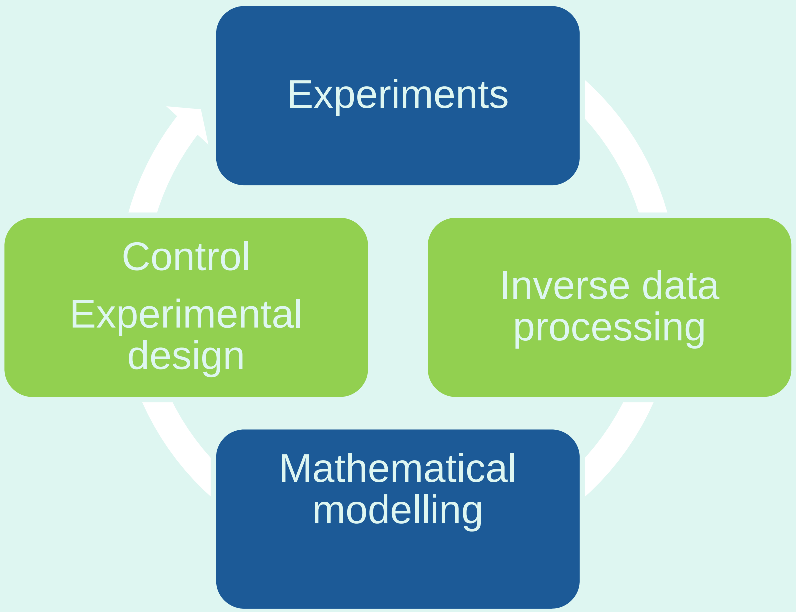
Experiments
Inverse data processing
Mathematical modelling
Control
Experimental design