Lecture 10: Linear Inverse Heat Conduction Problems
Two basic examples

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Introduction

- How to determine the time varying heat flux density entering a solid wall, from noisy data given by some temperature measurements inside (or outside) the wall, is a very standard Inverse Heat Conduction Problems (IHCP),

- The choice (in practice) of a numerical method for solving such problems will depend on the "complexity" of the model equations, and the "quality" of the measurements
    - Are the model equations linear or not?
    - What is the dimension and/or the shape of the spatial domain?
    - Which kinds of sensors? Their locations ? The output equations ?

...

- In any case, some specific difficulties are "expected", because IHCP are known to be ill-conditioned and regularized processes have to be developed for avoiding instable solutions due to noisy data, and/or biased models

Outline

- **Introduction**

- **Inverse Heat conduction in a semi infinite body**
  - the linear input/output model equation
  - Non regularized solutions – unstabilities
  - Regularized solution – the SVD method

- **Inverse Heat conduction in a plane wall**
  - the linear input/output model equation (single output)
  - Non regularized solutions – unstabilities
  - Effect of a biased model
  - Effect of a multi-output sensor
  - Splitting IHCP

- **Conclusion**

Semi-infinite heat conduction body
The model equations (see lecture n°2)

$$y_{mo}\ (t) = \int_0^{\infty} G\ (x_c,\ x,\ t)\ T_0\ (x)\ dx + \int_0^{t} Z\ (t - \tau)\ u(\tau)\ \mathrm{d}\tau$$

$$= y_{mo\ relax}\ (t) \qquad\qquad + \ y_{mo\ \ forced}\ (t)$$

$$G\ (x_c, x, t) = \frac{1}{2\sqrt{\pi\ a\ t}}\left[\exp\left(-\frac{(x_c - x)^2}{4\ a\ t}\right) + \exp\left(-\frac{(x_c + x)^2}{4\ a\ t}\right)\right]$$

$$Z\ (t) = \frac{1}{\sqrt{k\rho C\pi\ t}}\ \exp\left(-x_\mathrm{c}^2 / 4\,a\,t\right)$$

$$= K\sqrt{\frac{\tau}{t}}\exp(-\frac{\tau}{t})$$

Semi-infinite heat conduction body

$$Z^*(t^*) = Z(t^*)/K$$

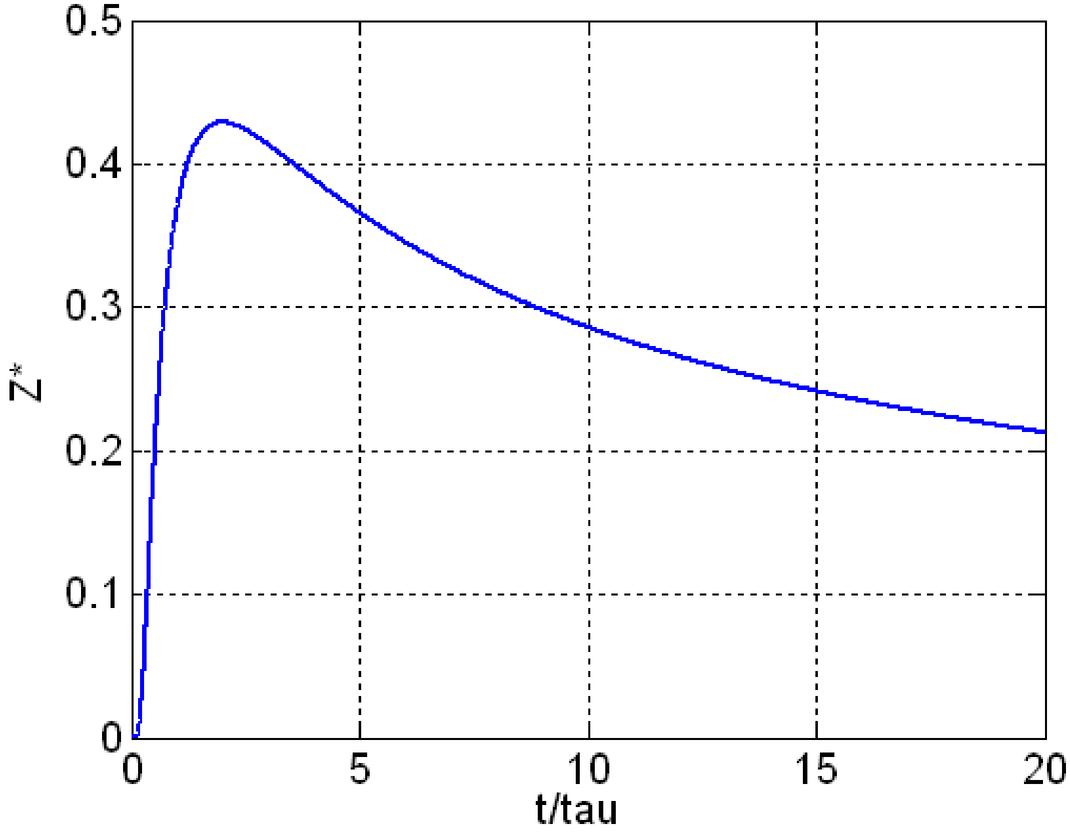


Figure 1 – The impulse output signal

$$\tau = \frac{x_c^2}{4a} \quad ;$$

$$t^* = \frac{t}{\tau}$$

and

$$K = \frac{2}{\rho c \; x_c \sqrt{\pi}}$$

Semi-infinite heat conduction body

The model equations

$$y_{mo}(\mathrm{t}_i) = \Delta t \sum_{j=1}^{i} Z\ (t_i - t_{j-1})\ u_j = \sum_{j=1}^{i} \mathrm{S}_{\mathrm{i\,j}}\ u_j$$

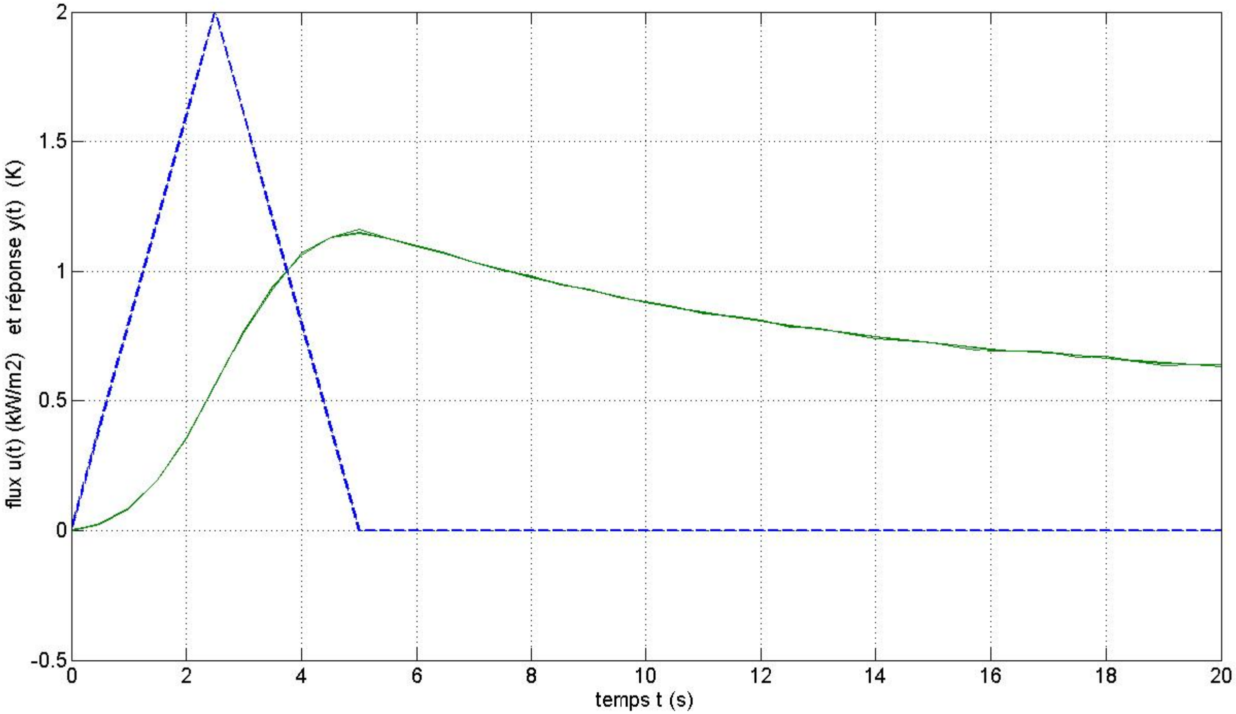
$$\mathrm{S}_{\mathrm{i\,j}} = \begin{cases} \Delta t\ Z\left((\mathrm{i} - j + 1)\ \Delta t\right) = z_{i-j+1}; & j = 1\ to\ i,\ i = 1\ to\ m \\ 0 & else \end{cases}$$

$$\mathbf{y}_{mo} = \mathbf{S}\ \mathbf{u}$$

$$\boldsymbol{S} = \begin{bmatrix} z_1 & 0 & & & & \\ z_2 & z_1 & 0 & & 0 & \\ z_3 & z_2 & z_1 & & & \\ z_4 & z_3 & z_2 & \ddots & 0 & \\ & & & \ddots & z_1 & 0 \\ z_m & z_{m-1} & z_{m-2} & & z_2 & z_1 \end{bmatrix}$$

Semi-infinite heat conduction body

Example of Input signal *u* (*t*) and output response *y(t)*



Numerical results computed with

$$\sigma = 0.005K$$

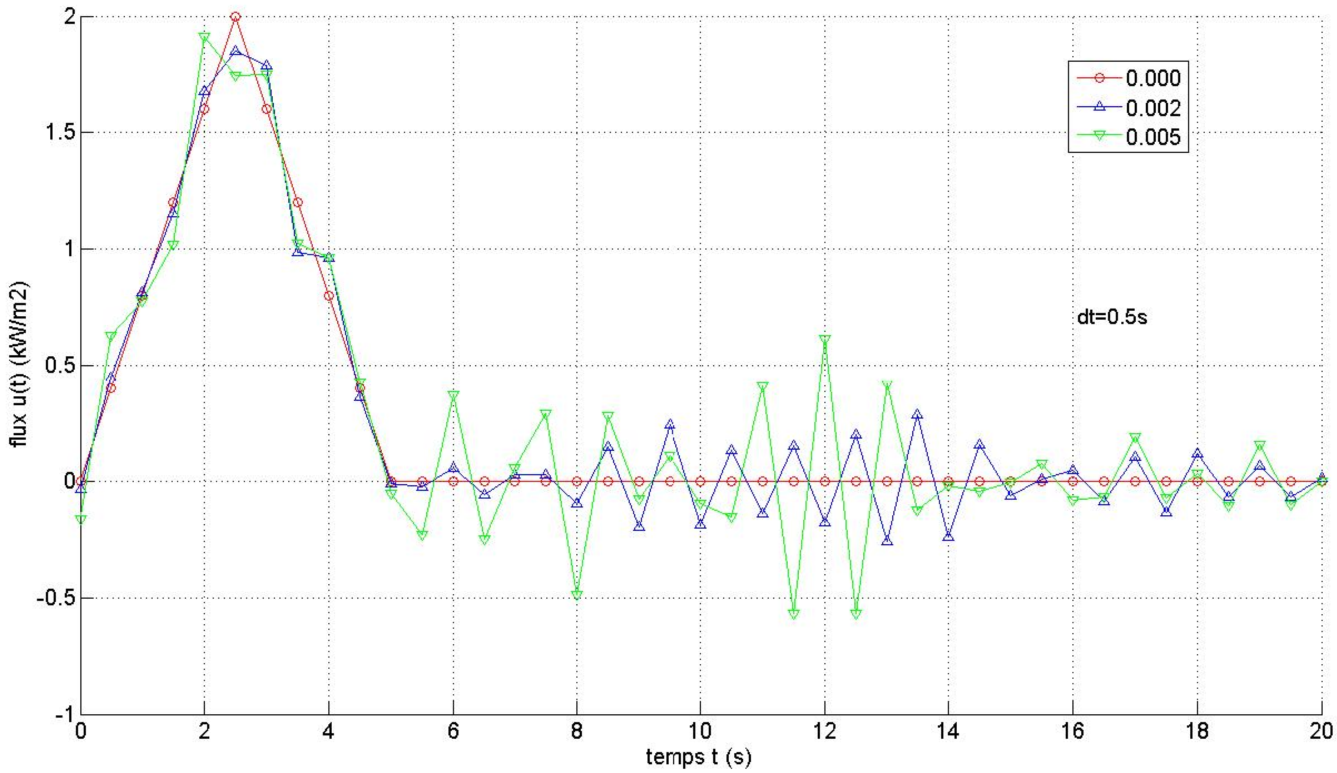
$$\Delta t = 0{,}5s$$

$$x_c = 2 \text{ mm} ; a = 10^{-6}\,\text{m}^2\text{s}^{-1}; \tau = 1 \text{ s} \quad \lambda = 1 \text{ W m}^{-1}\text{ K}^{-1}$$

$$\rho c = 10^6 \text{ J kg}^{-1}\text{ K}^{-1} \quad K = 0.564\ 10^{-3}\ \text{K m}^2\ \text{J}^{-1}$$

The IHCP in a semi-infinite body a non regularized solution

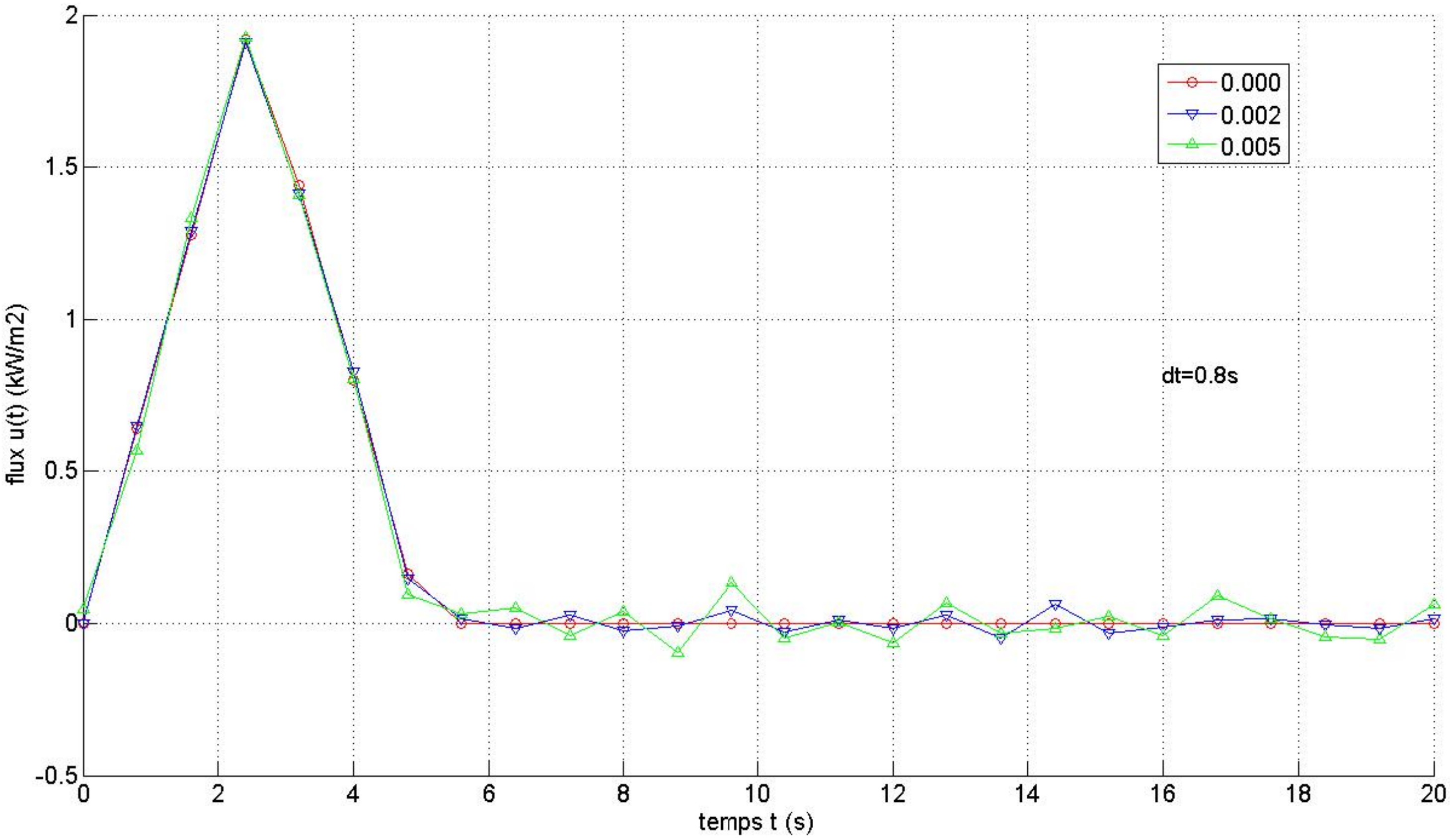
$$\hat{\mathbf{u}} = \mathbf{S}^{-1}\mathbf{y}$$



Estimated heat flux – cases a, b and c - Influence of the noise level on the computed heat flux - time step $\Delta t = 0{,}5$ s

The IHCP in a semi-infinite body a non regularized solution

$$\hat{\mathbf{u}} = \mathbf{S}^{-1}\mathbf{y}$$



Estimated heat flux – cases a, b and c - Influence of the noise level on the computed heat flux -

$$\Delta t = 0{,}8\text{s}$$

The IHCP in a semi-infinite body- Influence of the time step on the stability of the solution

by decreasing the time step,
the sensitivity coefficients of the Toeplitz matrix ***S*** goes to zero,
and the condition number grows exponentially

Δt	0.8	0.5	0.4	
Cond(S)	46,5	292	28420	

on each time step

$$\Delta u = \frac{1}{\Delta t}\int_{t_k}^{t_k+\Delta t} 800\ t\ \mathrm{d}t = 400\,\Delta t$$

The resulting increment on the output signal

will be "significant"

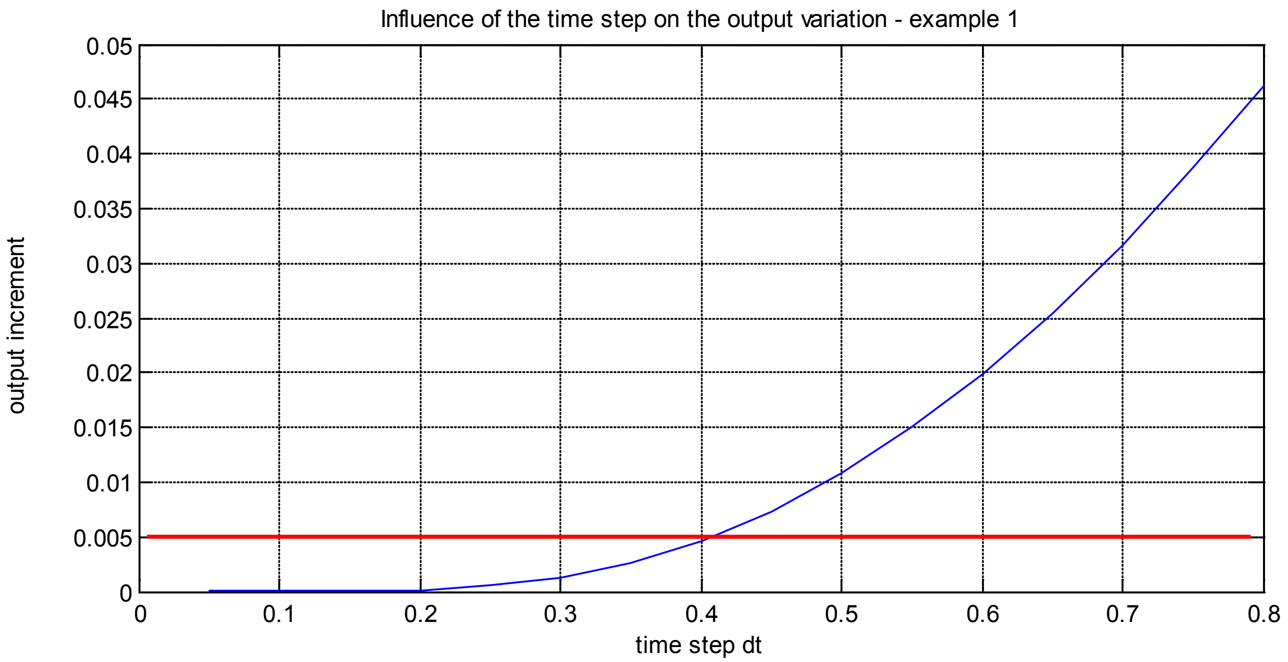
$$\Delta y \approx K\Delta t\sqrt{\frac{\tau}{\Delta t}}\ \exp(-\tau/\Delta t)\Delta u$$

only if this value is greater than the level noise

$$\Delta y > \sigma$$

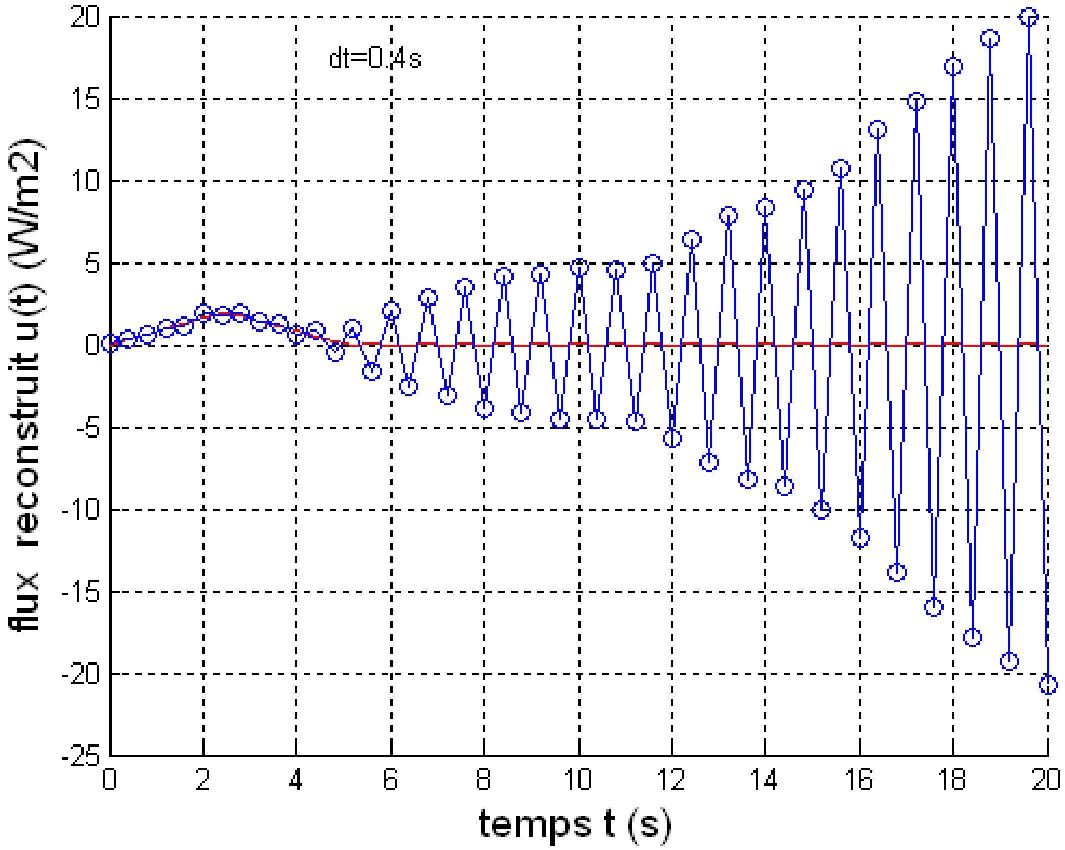
Influence of the time step on the stability of the solution

Δt	0.8	0.5	0.4	
Δy	46,3mK	10 mK	4,7mK	



The IHCP in a semi-infinite body a non regularized solution

$$\hat{\mathbf{u}} = \mathbf{S}^{-1}\mathbf{y}$$



$\sigma = 0{,}005$ K

$\Delta t = 0{,}4$ s

The IHCP in a semi-infinite body
a regularized solution - the SVD method

$$\boldsymbol{S} = \boldsymbol{U}\ \boldsymbol{W}\ \boldsymbol{V}^{T}$$

$\boldsymbol{U}, \boldsymbol{V}$ are *(m x m)* and *(n x n)* orthogonal matrices ; here m = n = 51

W is the matrix of the singular values $\{w_k\,, k = 1,..m\}$

The SVD regularized solution is then

$$\hat{\mathbf{u}}_r = \sum_{k=1}^{r<n} \frac{a_k}{w_k}\ \mathbf{V}_k \quad with \quad a_k = \mathbf{U}_k^T\ \mathbf{y}$$

The IHCP in a semi-infinite body
a regularized solution - the SVD method

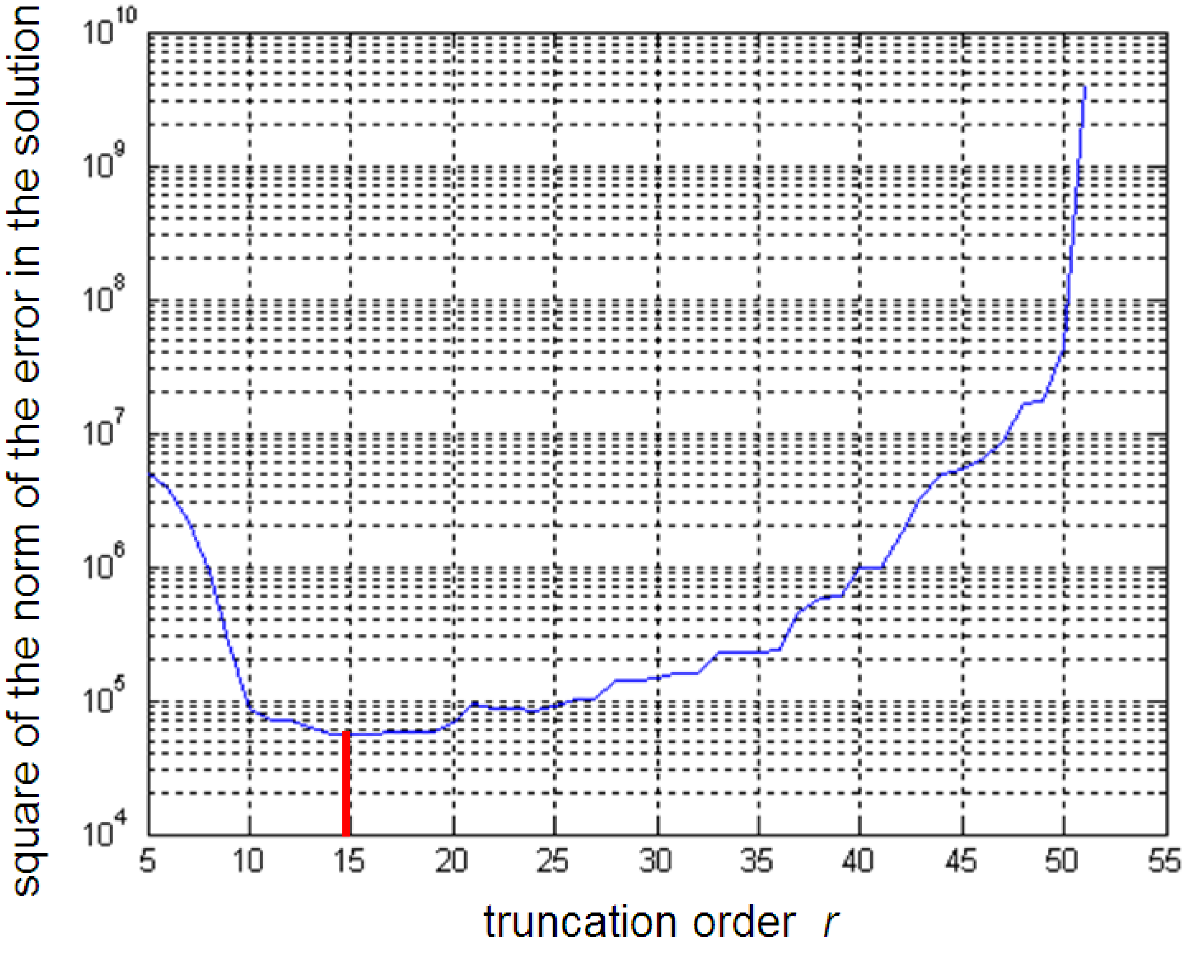
The truncation order *r* is used as the “tuning” parameter of the regularization process

The expected compromise between accuracy and stability will be fixed by some optimal value $0 < r < n$

We have to avoid:

- a too big error amplification , when $r \to n$
- a too large bias when $r \to 0$

The IHCP in a semi-infinite body
a regularized solution - the SVD method

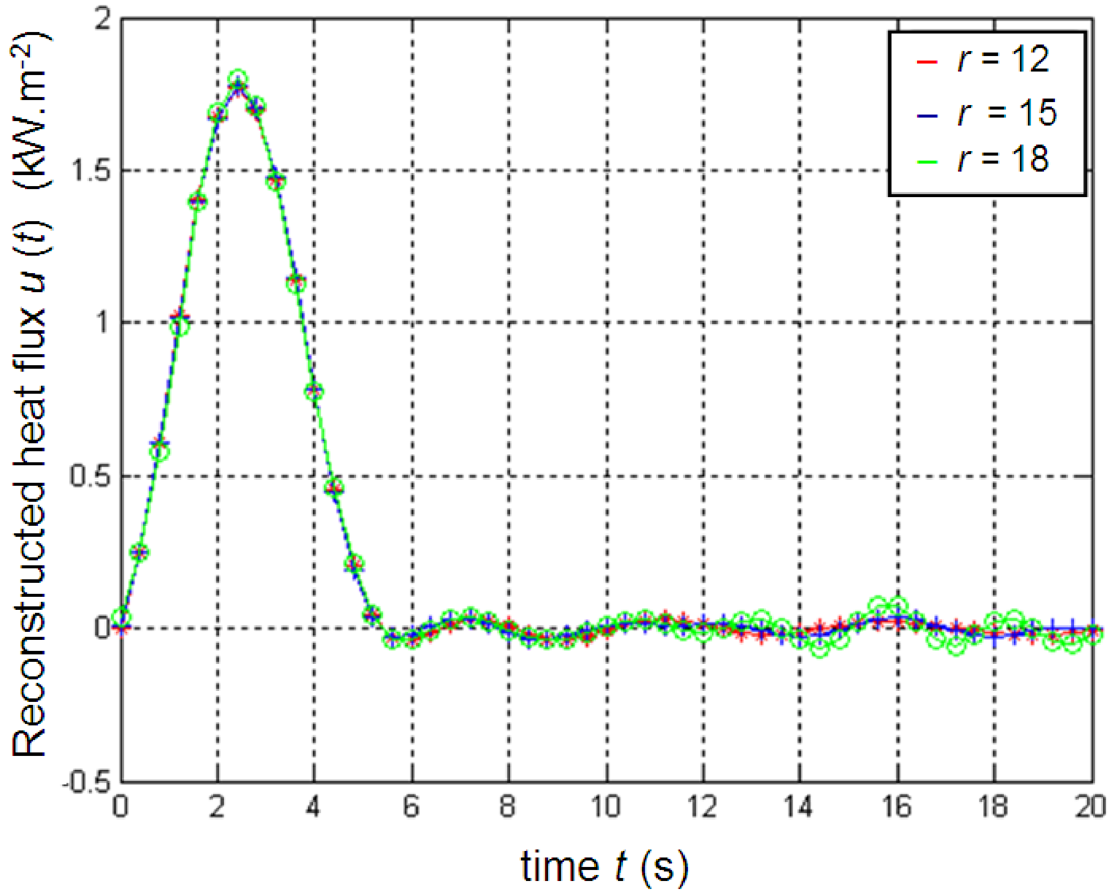


$\sigma = 0{,}005$ K

$\Delta t = 0{,}4$ s

$f(r) = \left\| \hat{\boldsymbol{u}}_r - \boldsymbol{u} \right\|^2$

The IHCP in a semi-infinite body a regularized solution - the SVD method



SVD Regularized solution computed with

$$r = 12, 15 \text{ et } 18$$

$$\Delta t = 0{,}4 \text{ s}$$

$$\sigma = 0{,}005 \text{ K}$$

The IHCP in a sem i-infinite body
a regularized solution - the SVD method

$$\mathbf{U} = \begin{bmatrix}\mathbf{U}^r & \mathbf{U}^c\end{bmatrix}\ ; V = \begin{bmatrix}V^r & V^c\end{bmatrix};\ \mathbf{W} = \begin{bmatrix}\mathbf{W}^r & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^c\end{bmatrix} \quad \text{and} \quad \mathbf{u} = \begin{bmatrix}\mathbf{u}^r \\ \mathbf{u}^c\end{bmatrix}$$

With $c = m - r$

*t*hen the error estimate can be put in the form

$$\boldsymbol{e}_u = \boldsymbol{V}^r \left(\boldsymbol{W}^r\right)^{-1} \left(\boldsymbol{U}^r\right)^T \boldsymbol{\varepsilon} - \boldsymbol{V}^c\ \widehat{\boldsymbol{u}}_c^*$$

$$\mathrm{E}\left(\boldsymbol{e}_u{}^T\ \boldsymbol{e}_u\right) = \sigma^2 \sum_{i=1}^{r} \frac{1}{w_k^2} + \sum_{k=r+1}^{m} \widehat{u}_k^{*\,2}$$

- The first term is directly linked to the variance of the measurement noise, it increases by increasing the truncation parameter *r*,
- and the second term depends only on the $c = m - r$ spectral components of the exact heat flux signal, which have been “lost” by truncation.

The IHCP in a sem i-infinite body
a regularized solution - the SVD method

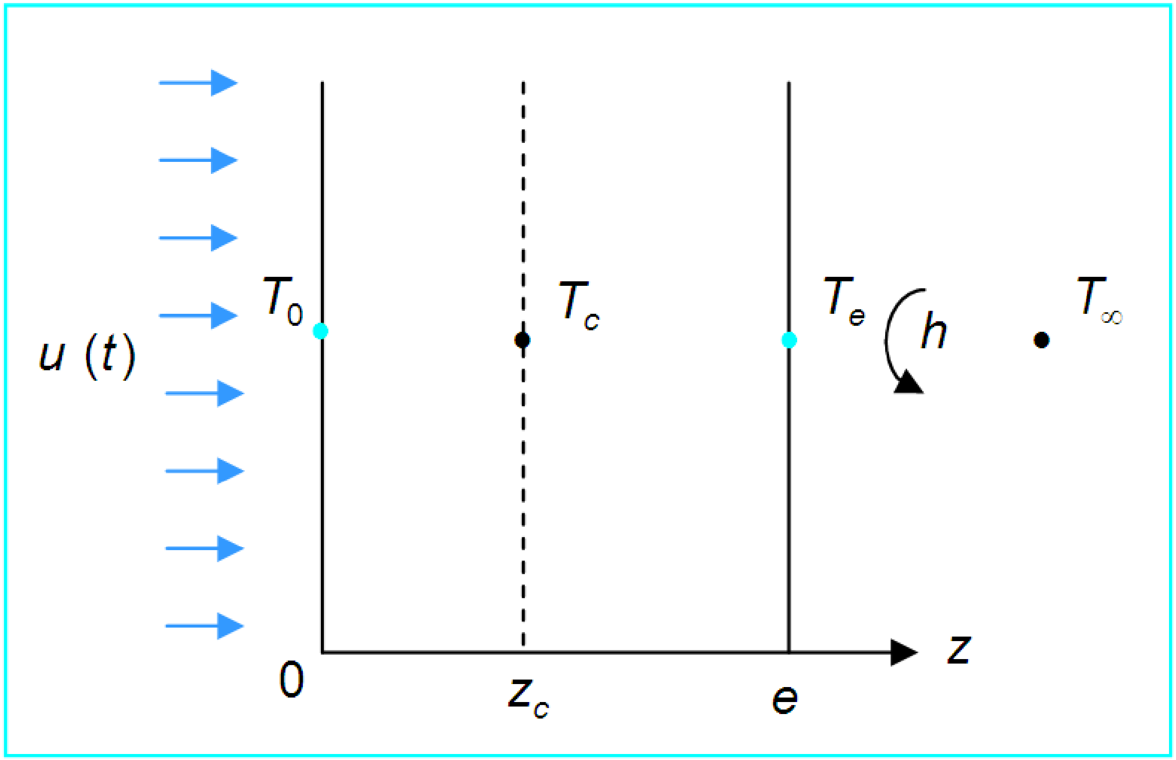
Conclusion : the ill-conditioness of the inverse heat conduction problem depends both

- ❑ on the mathematical model equations (singular values of **S)**
- ❑ and on the spectral values of the input signal to be determined

The compromise in choosing the truncation parameter *r* takes into account these both contributions.

Heat conduction in a plane wall

The model equations



Transient heat conduction in a plane wall

$$\frac{d\mathbf{T}}{dt} = \mathbf{A}\ \mathbf{T} + \mathbf{b}\ u(t)$$

$$\mathbf{T}\ (t = 0) = \mathbf{0}$$

$$T_0 = T_\infty = 0$$

$$\mathbf{y}_{mo}\ (t) = \mathbf{C}\ \ \mathbf{T}\ (t)$$

Heat conduction in a plane wall
Solution of the direct problem

$$\mathbf{T}(t) = \begin{bmatrix} T_1(t) & T_2(t) & \cdots & T_N(t) \end{bmatrix}^T \quad \text{avec} \quad T_i(t) = T(\mathrm{z}_\mathrm{i}, t)$$

$$\textit{and} \quad \mathrm{z}_\mathrm{i} = (i-1)\,\Delta \mathrm{z} \quad ; \quad \Delta \mathrm{z} = \frac{e}{N-1}$$

$$\boldsymbol{A} = \frac{a}{(\Delta z)^2} \begin{bmatrix} -2 & 2 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & 0 \\ \vdots & & \ddots & & \vdots \\ & & & -2 & 1 \\ 0 & 0 & \cdots & 2 & -2(1+Bi) \end{bmatrix} \quad \text{et} \quad \boldsymbol{b} = \frac{2}{\rho c\,\Delta z} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \qquad Bi = h\,\Delta z / \lambda$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 \end{bmatrix} \text{ where } C_i = 0 \text{ si } i \neq i_c$$

$$\boxed{\mathrm{y}_{mo}(t) = \mathbf{C} \int_0^t \mathbf{exp}\,(\mathbf{A}\,(t-\tau))\,\mathbf{b}\,u(\tau)\,\mathrm{d}\tau}$$

Heat conduction in a plane wall
Discrete Solution of the direct problem

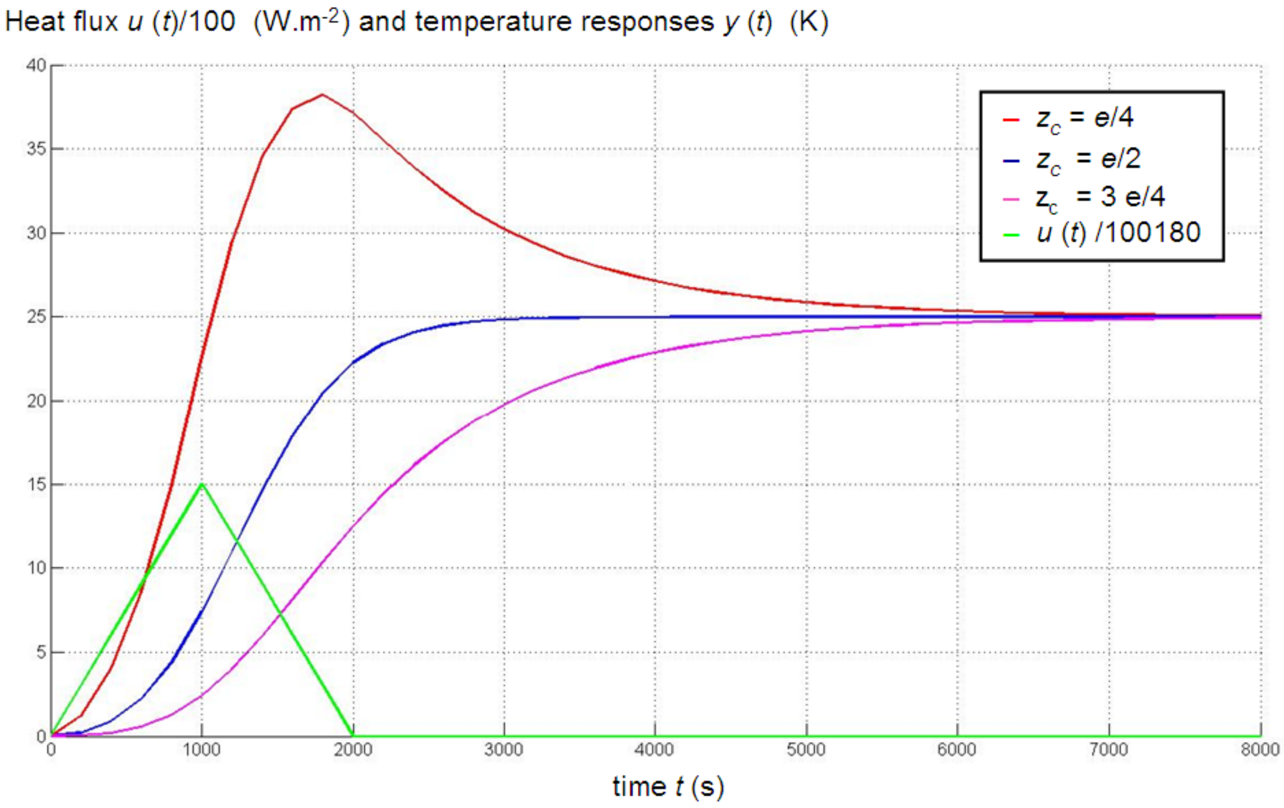
$$u(t) = \sum_{j=1}^{n} u_j \; f_j \; (t); \quad \left\{ f_j(t_k) = \delta_{ik} \right\}$$

$$\mathrm{y}_{mo} \; (t_k) = \sum_{j=1}^{n} S_{k\,j} \; u_j;$$

$$S_{k\,j} = \mathbf{C} \int_0^{t_k} \mathbf{exp} \; (\mathbf{A} \; (t_k - \tau)) \; \mathbf{b} \; f_j \; (\tau) \, \mathrm{d}\tau, \quad k = 1,..,m$$

$$\boxed{\mathbf{y}_{mo} = \mathbf{S} \; \mathbf{u}}$$

Heat conduction in a plane wall
Example of Numerical results



$$\boldsymbol{y} = \boldsymbol{y}_{mo} + \boldsymbol{\varepsilon}$$

$$N = 21 \text{ nodes}$$

$$\Delta t = 200s$$

$$nt = 40$$

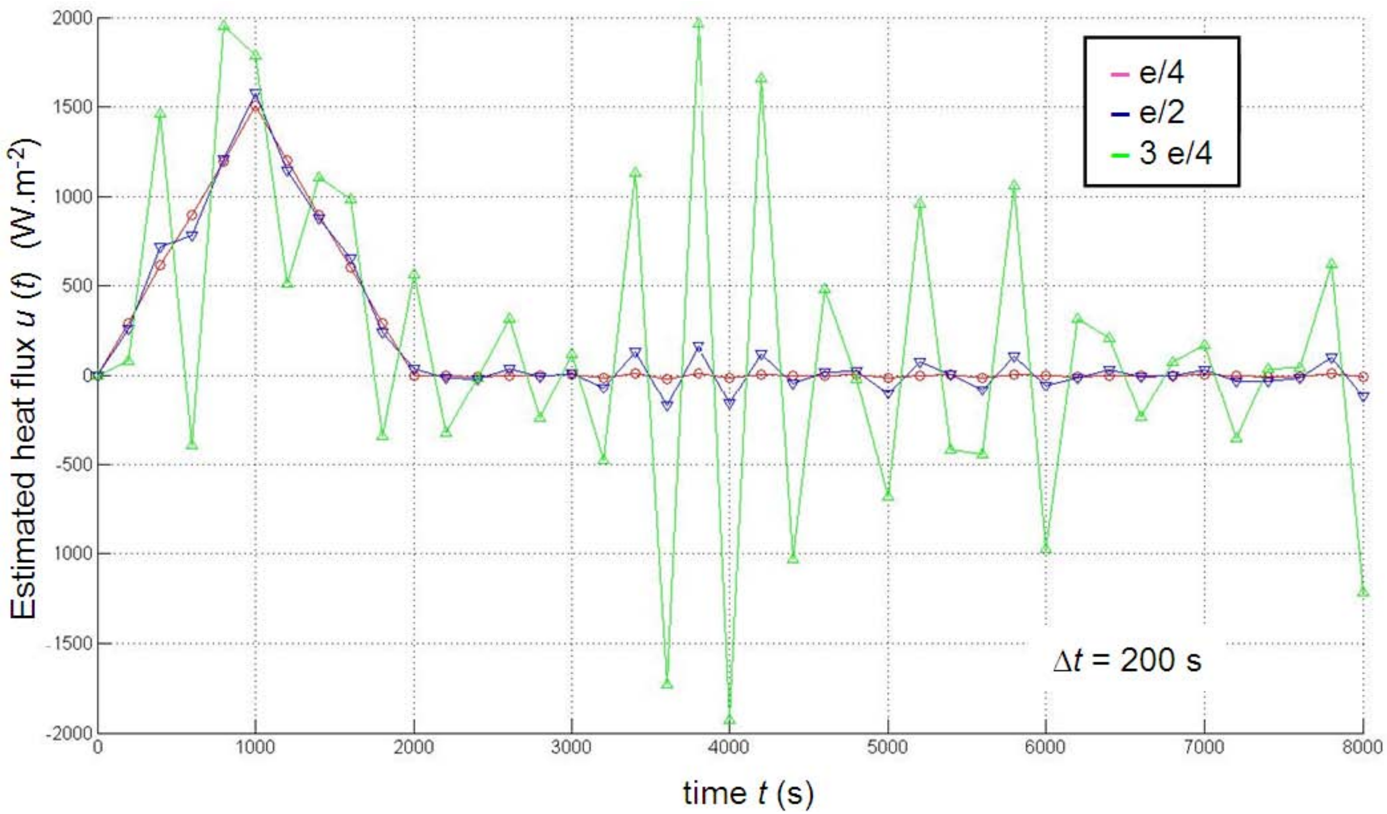
$$\sigma = 0{,}02\ K$$

$$e = 0{,}05 \text{ m}; \ \lambda = 0{,}3 \text{ Wm}^{-1}\text{K}^{-1}; \ \rho c = 1.2\ 10^6 \ \text{Jm}^{-3}\text{K}^{-1}; h = 0Wm^{-2}K^{-1}$$

$\hat{u}$

Inverse Heat conduction in a plane wall
non regularized solution

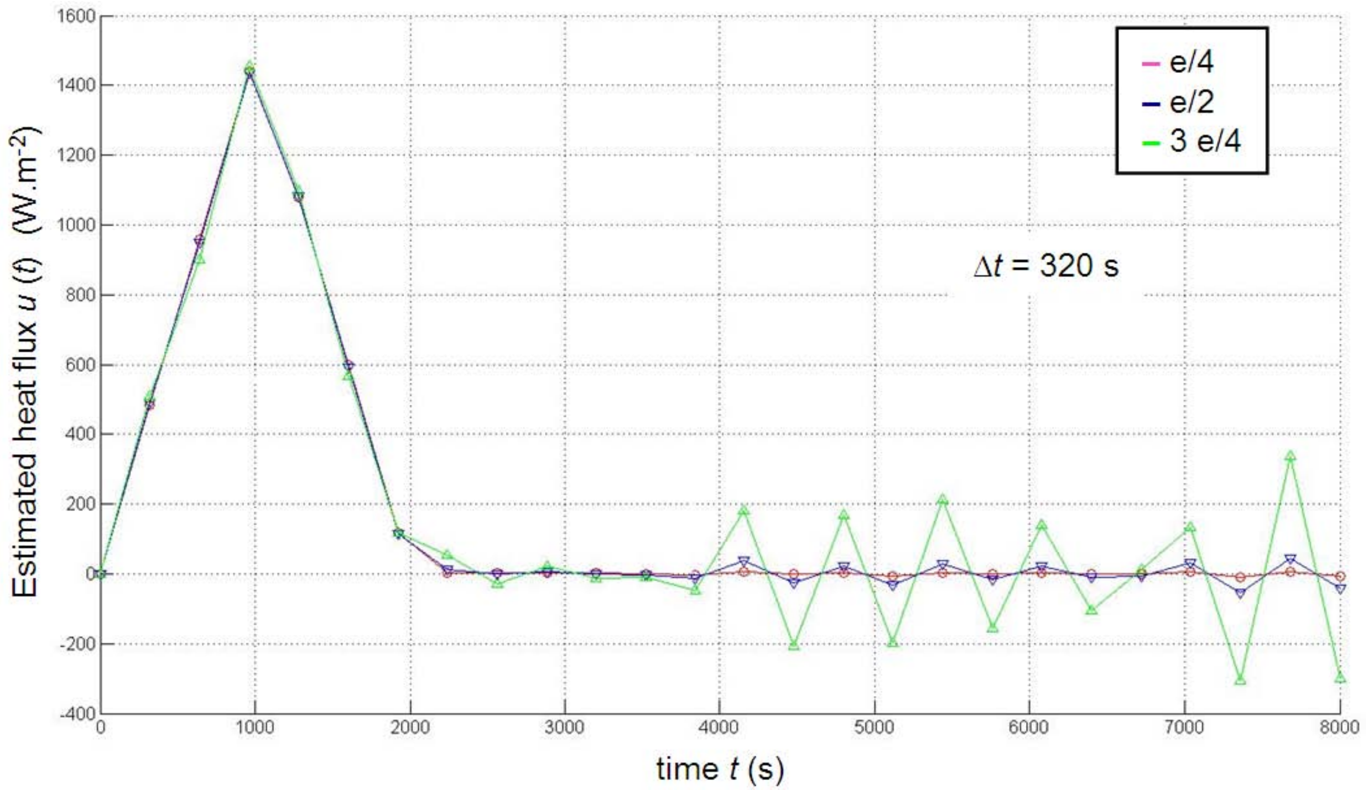
$$\hat{\mathbf{u}} = \mathbf{S}^{-1}\,\mathbf{y}$$



Estimated heat flux – Influence of the sensor location $\sigma = 0{,}02\ K$

Inverse Heat conduction in a plane wall
non regularized solution

$$\hat{\mathbf{u}} = \mathbf{S}^{-1}\mathbf{y}$$



Estimated heat flux – Influence of the sensor location

$\Delta t = 320$ s

$\sigma = 0{,}02\ K$

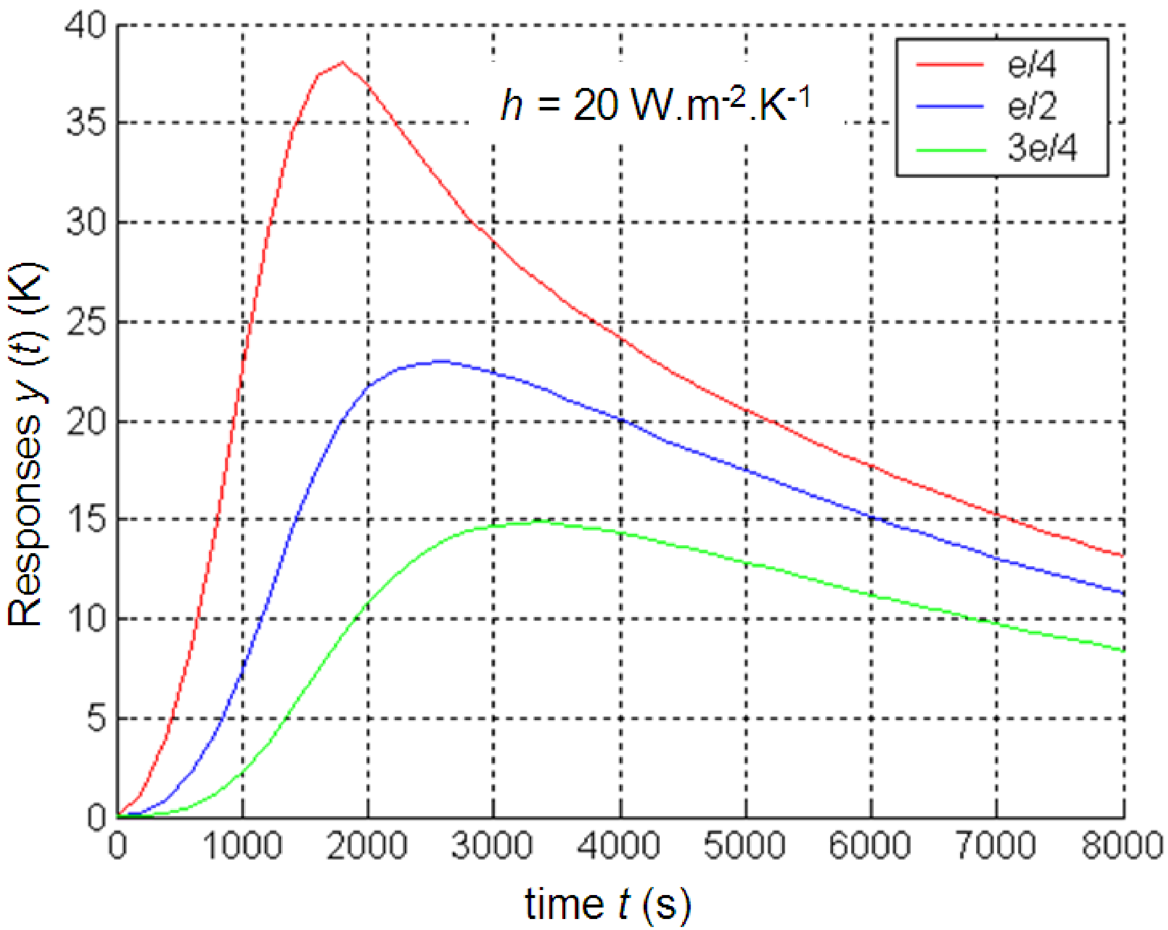
Inverse Heat conduction in a plane wall
non regularized solution

z_c	$e/4$	$e/2$	$3\,e/4$
$\Delta t = 320$ s	730	8700	$1{,}0244\ 10^5$
$\Delta t = 200$ s	343*	2349	$1{,}7\ 10^4$

Condition number of the matrix ***S*** - IHCP in a plane wall

As in the previous example, by using the SVD approach,
- regularized solutions could be easily computed
- and the same analysis of the estimation error could be done

Inverse Heat conduction in a plane wall *solution with a biased model*

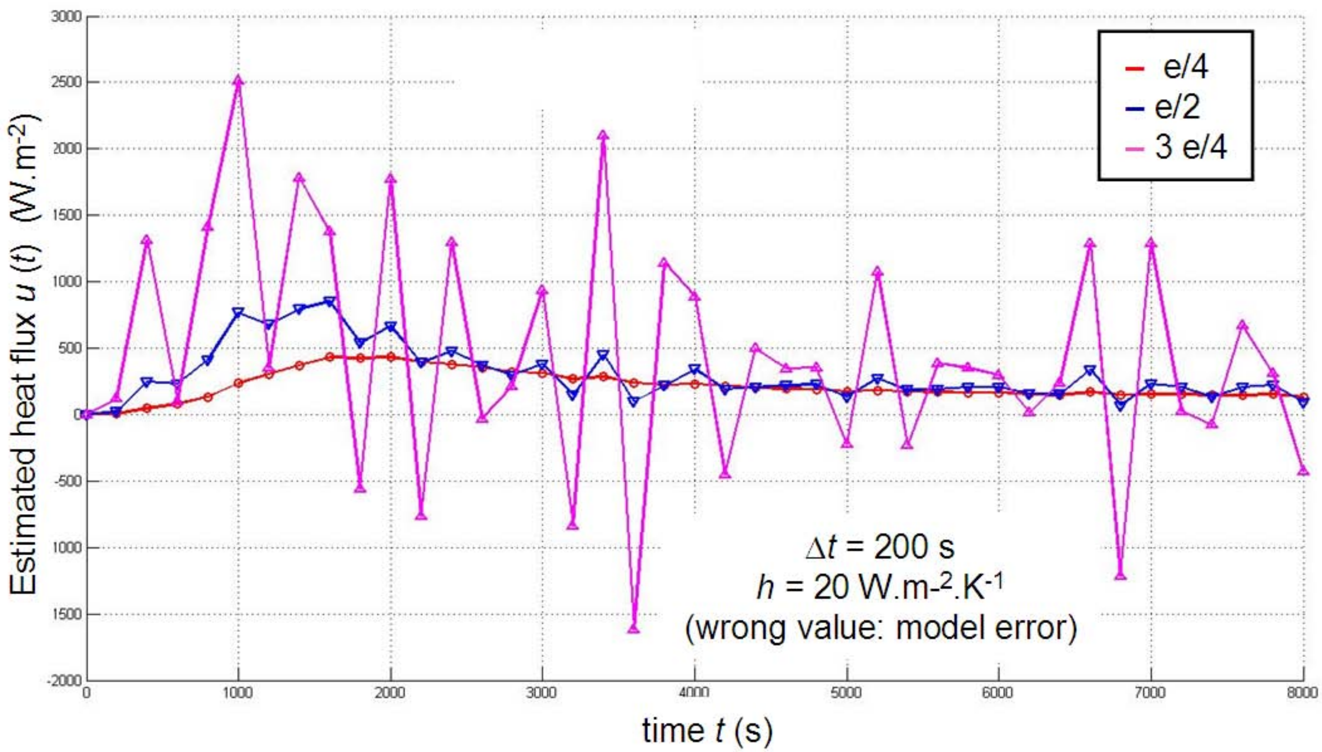


Output signal computed with
h = 20 W m^{-2} K^{-1}

(instead of h = 0)

The Influence of this parameter on the output signal becomes more and more significant when the sensor is located closer to the boundary $x = e$

Inverse Heat conduction in a plane wall *solution with a biased model*



The numerical inversion process is performed on the original noisy output data

(with *h = 0)* but with a model error (*h* = 20 W m^{-2}K^{-1}), included in the matrix ***S.***

Inverse Heat conduction in a plane wall *solution with a biased model*

The influence of the sensor location is clearly illustrated.

There is a systematic error $b_u(t) = \hat{u}_{h=20}(t) - u_{h=0}(t)$

between the solutions computed with the biased and the exact models.

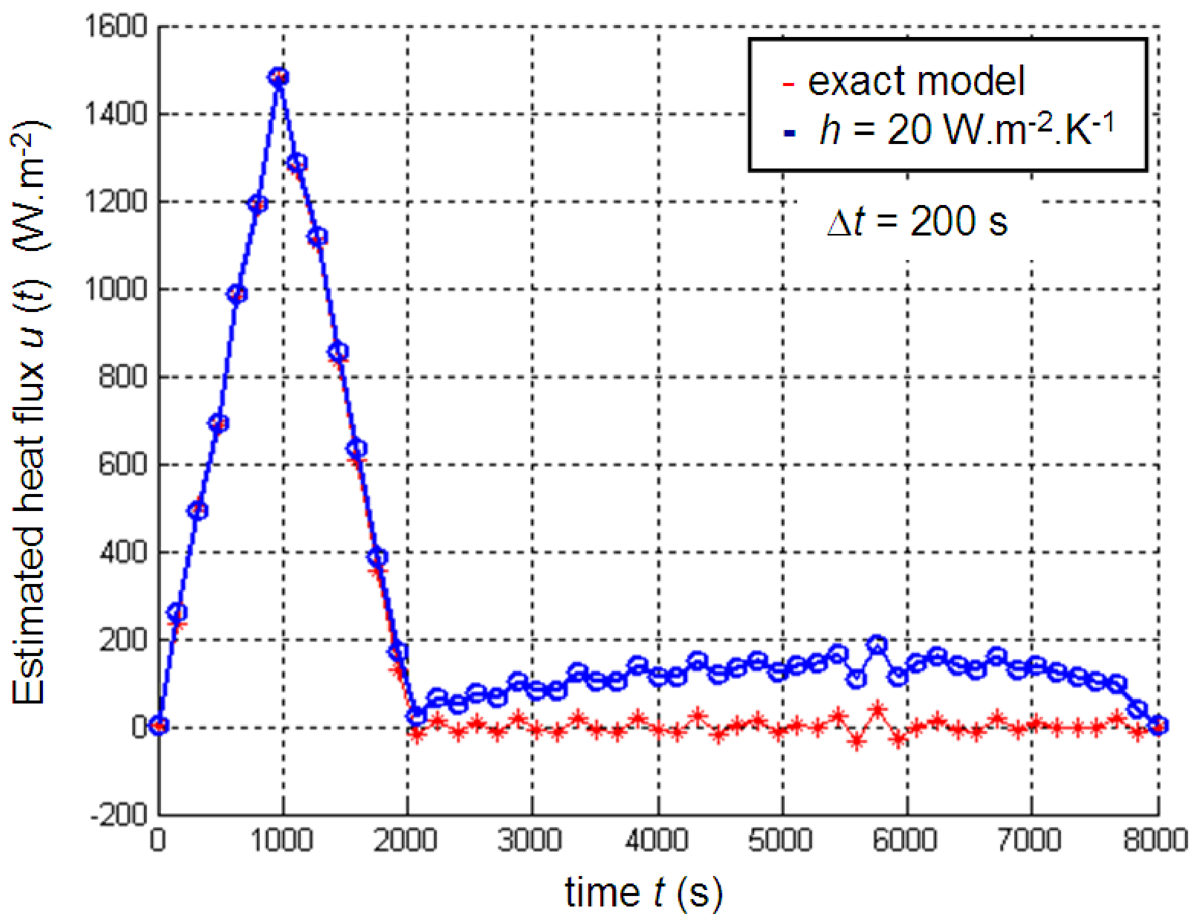
The mean value of this bias $b_u(t)$

is evident at the end of the time interval.

)

Inverse Heat conduction in a plane wall *solution with a biased model* *Effect of a multi-output sensor*

$$\hat{\mathbf{u}}_{OLS} = (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{y}$$

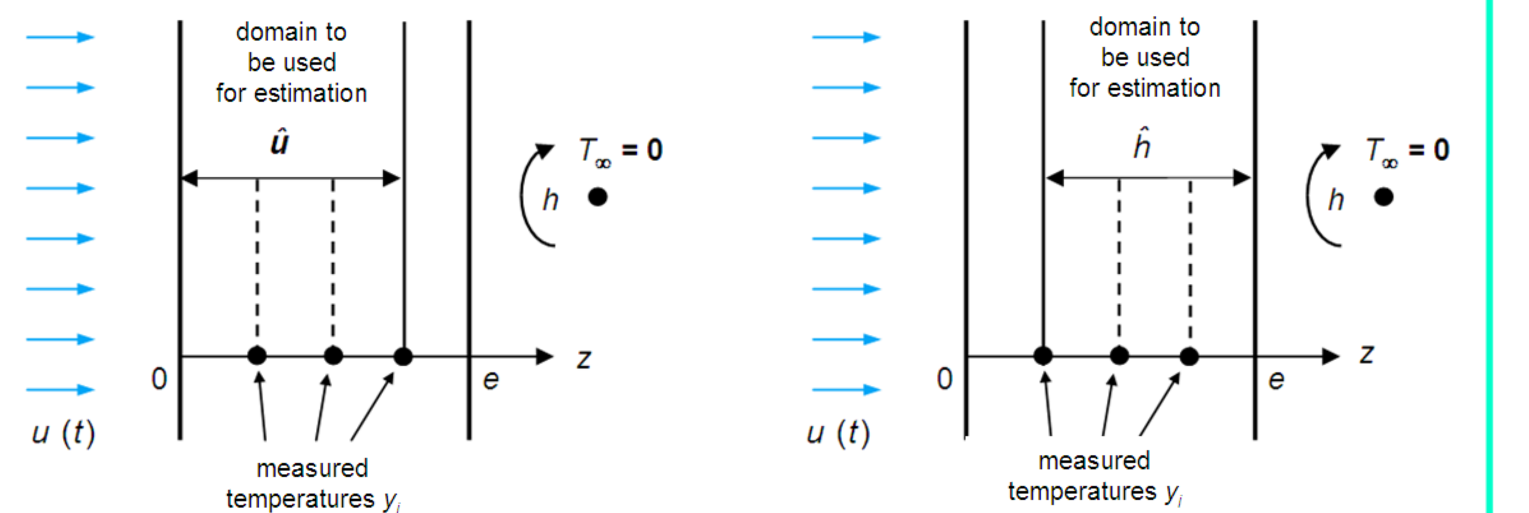


solutions obtained with two sensors located at $z = e/4$ and $z = e/2$ with noisy data

$\sigma = 0{,}02\ K$

$\Delta t = 200\ \text{s}$

Inverse Heat conduction problems in a plane wall



a – Flux estimation **b** – Estimation of heat transfer coefficient

Splitting the Inverse heat conduction problems for a plane wall case

- **the semi-infinite solid** = simple geometry which provides a useful idealization for many practical situations.
  - A numerical solution of this IHCP can be easily investigated, thanks to the linearity of the model equation.
  - The instabilities of the non regularized solutions due to: noise level, sensor location, time step ..., can be analyzed and illustrated.
  - The SVD method is a powerful approach to master the regularized solutions.
- the linear **heat conduction problem in a plane wall,**
  - after a standard discretization of the spatial variable
  - similar analysis of the instabilities of the IHCP solutions, can be investigated
  - the influence of a biased model, with/without a multi-sensor output model is easy to illustrate

References

[1] F P Incropera, D P DeWitt, *Fundamentals of Heat and Mass Transfer*, (IVth edition) John Wiley and Sons, New York, 1996

[2] D Maillet, Y Jarny and D Petit, *Problèmes inverses en diffusion thermique*, Techniques de l'Ingénieur", BE 8 266, Editions T.I., Paris, 2010

[3] Y Jarny and H Orlande, « *Adjoint Methods »*, in "*Thermal Measurements in Heat Transfer*", CRC Press, 2011